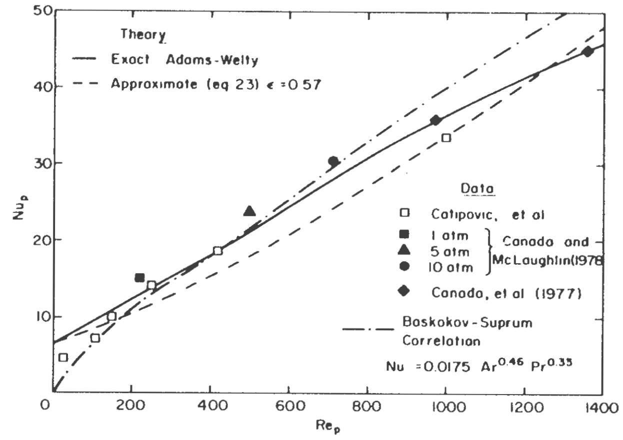

Fig. 2 Total Nusselt number for horizontal tube

results within about 10 percent of the numerical calculation. Note that values for the region $z/D < 0.3$ are not shown because a bubble is normally attached there.

The total Nusselt number for an immersed tube can be approximated using the model by neglecting voidage variations so that the gas velocity is

$$Q_g^+ \simeq \frac{2}{\epsilon}\sin\theta \tag{22}$$

where θ is measured from the lower stagnation point. Then, with $\beta_T \simeq 0.13u'\sqrt{\mathrm{Re}_p/\epsilon}$, integration of equation (12*b*) gives

$$(\mathrm{Nu}_p)_{\mathrm{Total}} \simeq \frac{\left(0.0917 + 0.154\left(\frac{S_p}{r_p} - 1\right)\right)\mathrm{Re}_p}{\epsilon S_0 N_{Ho}{}^2\left(\frac{S_p}{r_p}\right)\left(1.405 + 5.6e^{-0.0849u'\sqrt{\mathrm{Re}/\epsilon}}\right)} + \frac{2\,S_0}{(S_p/r_p)}\left(3.81 - 1.6\left(\frac{S_p}{r_p} - 1\right)\right) \tag{23}$$

Now, considering air (Pr = 0.7) low temperature operation (so that $S_0 = 1.0$), turbulence intensity, $u' = 0.2$, and using the correlation of Wen and Yu [5] for Re_p with $\epsilon = 0.57$ (the average surface value) the variation in total Nusselt number shown in Fig. 2 is obtained. Shown with the approximation are exact numerical results, an experimental correlation developed by Baskakov and Suprum [6] and experimental data obtained by Catipovic [3] and Canada [7] for operation near minimum fluidizing conditions. The major source of error in the approximation is the assumption of constant voidage, but the error is still at most 15 percent for the range of parameters indicated.

Conclusion

The approximate Nusselt number obtained using equation (13) provides a reasonable alternative to the numerical procedure developed by Adams and Welty [1] for local gas convection dominant Nusselt number. Integration of the approximate formula for heat transfer to an immersed horizontal tube with constant voidage provides results which do not match exact results with variable voidage but are within about 15 percent of those results and experimental data.

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Application of the Integral Method to Two-Dimensional Transient Heat Conduction Problems

W. W. Yuen[1] and R. A. Wessel[1]

Nomenclature

F = function defined by equation (8)
t = time
u = transformed coordinate defined by equation (4)
v = transformed coordinate defined by equation (4)
x = coordinate
y = coordinate
α = thermal diffusivity
δ = one-dimensional penetration depth
$\eta = u/\delta$
θ = temperature

Introduction

In the field of heat transfer, the integral method is one of the most powerful solution techniques. For practical engineering applications, the method is used to generate approximate closed-form solutions. The effect of system parameters and the importance of various dimensionless groups can be readily demonstrated. For many numerical computations, the integral method is used to generate small-time solutions which otherwise are very difficult to obtain. Review of the integral method and its application to both linear and nonlinear heat transfer problems is available in the literature [1].

The objective of this work is to show that the integral method can be generalized to apply to the two-dimensional transient conduction problem. The problem of a rectangular corner with uniform initial temperature is solved as an illustration.

Mathematical Formulation

The physical model and its associated coordinate system for the rectangular-corner problem is shown in Fig. 1(*a*). For simplicity, all thermal properties are assumed to be constant. The conduction equation is

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{1}$$

with θ being the temperature and α the thermal diffusivity. The initial condition is

$$\theta(x,y,t) = 0 \qquad t < 0, \qquad \text{all } x,y \tag{2}$$

[1] Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, Calif. 93106.

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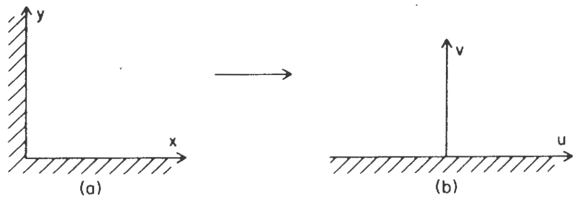

Fig. 1 The original and transformed coordinate system for the present problem

The boundary conditions for the two cases considered in the present work are

$$\begin{aligned} \theta(0,y,t) &= 1 \quad t \geqq 0, \quad y \geqq 0 \\ \theta(x,0,t) &= 1 \quad t \geqq 0, \quad x \geqq 0 \end{aligned} \tag{3}$$

The above initial and boundary conditions are obviously normalized for simplicity. Since the problem is linear, solutions to problems with unnormalized conditions can be readily generated by superposition of the present results.

As they are, equation (1) together with the initial and boundary conditions are practically impossible to be solved by the integral method. Physically, application of the integral method requires the existence of a characteristic length such that beyond this length, the influence of the wall can be neglected. While it is clear that at any given time there exists a certain interior region of the rectangular corner which is relatively unaffected by the heated wall, the definition of a "length" for the problem under the present geometry appears to be extremely difficult, if not impossible.

This difficulty can be alleviated, however, if the following coordinate transformation is utilized.

$$u = x^2 - y^2 \tag{4}$$

$$v = 2xy \tag{5}$$

Mathematically, equations (4) and (5) represent the familiar conformal transformation [3] which is often used in the solution of the steady state conduction problem. Physically, this transformation maps the rectangular corner into the upper half plane as shown in Fig. 1(b). The curves u = const. and v = const. can be interpreted as the flux line and potential line for the rectangular corner in potential theory. In the transformed coordinate, equations (1, 2), and (3) become

$$4(u^2 + v^2)^{1/2}\left[\frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2}\right] = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} \tag{1a}$$

$$\theta(u,v,t) = 0 \qquad t < 0, \qquad \text{all } u,v \tag{2a}$$

$$\theta(u,0,t) = 1 \qquad t \geqq 0 \tag{3a}$$

It is interesting to note that the transformed problem can now be interpreted as the problem of transient conduction in a semi-infinite medium with a spatially-dependent thermal diffusivity.

While a complete solution of equation (1a) together with its initial and boundary conditions is just as difficult as it is for the original problem, the transformed problem has the advantage that it is more adaptable for the integral method. Physically, it is reasonable to assume that for each value of u at time t, there exists a penetration depth $\delta(u,t)$ such that $\theta = 0$ for all $v > \delta$. As in the traditional integral method for one-dimensional transient conduction problem, a temperature profile can be assumed at each value of u. A differential equation for $\delta(u,t)$ can be generated by an integration of equation (1a).

Results

Similar to the one-dimensional transient conduction problem, the temperature profile for the case with constant wall temperature is first assumed to be a second degree polynomial as follows

$$\theta(u,v,t) = \left(1 - \frac{v}{\partial(u,t)}\right)^2 \tag{6}$$

Substitute equation (6) into equation (1a) and integrate over v from 0 to δ, the following equation is resulted.

$$\frac{1}{3}\frac{\partial^2\delta}{\partial u^2} + \frac{2}{\delta} = \frac{1}{4\alpha}\frac{\partial F(\delta,u)}{\partial\delta}\frac{\partial\delta}{\partial t} \tag{7}$$

The function $F(u,\delta)$ in the above equation is defined as

$$F(u,\delta) = \left(1 - \frac{1}{2}\eta^2\right)\ln\left[\frac{1 + (1+\eta^2)^{1/2}}{\eta}\right] - \frac{3}{2}(1+\eta^2)^{1/2} + 2\eta \tag{8}$$

where $\eta = u/\delta$. The boundary conditions and initial condition for equation (7) are

$$\frac{\partial\delta}{\partial u}(0,t) = 0 \tag{9}$$

$$\lim_{u\to\infty} \delta(u,t) = (48u\alpha t)^{1/2} \tag{10}$$

$$\delta(u,0) = 0 \tag{11}$$

Physically, the problem is symmetric about the origin $u = 0$. Equation (7) is thus needed to be solved only for cases with $u > 0$. Equation (9) is the result of such symmetry. Equation (10) is based on the fact that in the limit of large u, the present result should reduce to the one-dimensional case which yields the following expression for the approximate temperature profile [4].

$$\theta(y,t) = \left(1 - \frac{y}{(12\alpha t)^{1/2}}\right)^2 \tag{12}$$

While an exact numerical solution to equation (7) is not difficult to obtain, a closed-form early-time solution can be readily generated by assuming that the second derivative of δ is negligible. Equation (7) can then be integrated to yield

$$\begin{aligned} \frac{8\alpha t}{\delta} = -\eta^2 \ln\left[\frac{1+(1+\eta^2)^{1/2}}{\eta}\right] + \frac{\eta}{2}\ln \\ \times \left[\frac{2(1+(1+\eta^2)^{1/2} - \eta)\eta^4}{(1+(1+\eta^2)^{1/2}+\eta)(\eta + (\eta^2+1)^{1/2})^3}\right] \\ + 1 + \frac{\eta^2}{[1+(1+y)^2]^{1/2}} \end{aligned} \tag{13}$$

It is interesting to note that equation (6), utilizing the value of δ generated by equation (13), already yields temperature distributions compared favorably with the exact solution, particularly in the limit of large u. The worst accuracy appears to occur at $u = 0$. Equation (13) gives

$$\delta(0,t) = 8\alpha t \tag{14}$$

Equation (6) becomes

$$\theta(0,v,t) = \left(1 - \frac{v}{8\alpha t}\right)^2 \tag{15}$$

Note that since the line $u = 0$ in the u-v plane corresponds to the line $x = y$ in the x-y plane, the temperature profile is only a function of x. The exact expression of the temperature profile is available [2]. It is

$$\theta(x,t) = 1 - \left[\operatorname{erf}\frac{x}{2(\alpha t)^{1/2}}\right]^2 \tag{16}$$

Compared to equation (16), equation (15) is not very accurate, except in the limit of small x and large t. But it still yields the correct qualitative behavior of the temperature profile.

Since equation (7) is of the same general form as a one-dimensional transient conduction problem, its solution can be readily obtained by any standard numerical computation routine such as the tri-diagonal algorithm. Results of temperature profiles generated by the present integral method compare well with the available exact solution [2]. For the least accurate case at $u = 0$, approximate temperature profiles at two different times are compared with the exact solution in Fig. 2. The agreement is quite acceptable. The numerical solution also suggests that for large values of αt ($\alpha t > 1$), the penetration depth

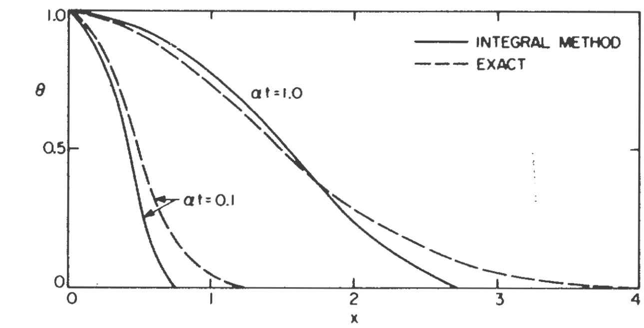

Fig. 2 Comparison between temperature profiles generated by the integral method and the exact result for the problem with a constant wall temperature

at $u = 0$ can be approximated as

$$\lim_{\alpha t \to \infty} \delta(0,t) \to 16\alpha t \tag{17}$$

The temperature profile at $u = 0$ becomes

$$\theta(x,t) = \left(1 - \frac{x^2}{8\alpha t}\right)^2 \tag{18}$$

Equation (18) represents a fairly accurate approximation for equation (16).

Conclusion

Utilizing a simple conformal transformation which maps the region of interest onto the upper half plane, the integral method is generalized to apply to the two-dimensional transient conduction problem. Transient heat conduction in a rectangular corner subjected to a constant temperature boundary conditions is solved as an illustration of the method. Results compare well with the available exact result and they are generated with little effort.

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The Temperature Distribution of a Sphere Placed in a Directed Uniform Heat Flux

K. A. Werley[1] and J. G. Gilligan[1]

Nomenclature

α = thermal diffusivity $\equiv \dfrac{k}{dc_p}$

β, γ = eigenvalues

c_p = specific heat

d = density

F = heat flux

J = Bessel function

k = thermal conductivity

λ, μ = eigenvalues

m, n = summation indices

P = Legendre polynomial

ρ = radial position in sphere

r = dimensionless position $\equiv \rho/R$

R = radius of sphere

τ = time

t = dimensionless time $\equiv \dfrac{\alpha\tau}{R^2}$

θ = angle

T = temperature

T_0 = initial temperature

Introduction

The purpose of this Technical Note is to solve analytically the time-dependent heat conduction problem of a sphere, with uniform initial temperature, located in a uniform field of directed heat flux. To the authors' knowledge, this solution has not been displayed, but is of interest in the area of magnetic thermonuclear reactor engineering. It has been shown that a "rain" of high speed liquid or solid spheres (e.g., Lithium) can effectively remove over a magnitude larger heat flux than a stationary solid wall while simultaneously gettering large fluxes of particles, thereby acting as a vacuum pump [1–3]. Thus, the importance of this concept requires that an accurate description of the temperature distribution be known so that vaporization rates, thermal stresses, and particle trapping efficiencies can be calculated.

Problem Description

The problem of a sphere with an initial uniform temperature, T_0, located in a field of directed uniform heat flux, F, is described by the heat conduction equation in spherical coordinates (see Fig. 1):

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial T}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) = \frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

$$\times\ 0 \le \rho \le R; 0 \le \theta \le \pi; 0 \le \tau \tag{1}$$

It can be shown with simple geometrical arguments, that the directed uniform heat flux transforms to a surface flux which varies as the cosine distribution over the front of the sphere. Thus the initial and boundary conditions can be written:

$$T(\rho, \theta, 0) = T_0 \tag{2}$$

$$T(\rho, \theta, \tau) = \text{finite, for arbitrary } \theta \text{ and } \rho \tag{3}$$

$$\frac{\partial T}{\partial \rho}(R, \theta, \tau) = \begin{cases} \dfrac{F}{k}\cos\theta, & 0 \le \theta \le \dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2} \le \theta \le \pi \end{cases} \tag{4}$$

The above set of equations describes the temperature distribution in the sphere provided that the following four assumptions are valid: (*a*) The material has constant, uniform properties (k, α, d, c_p). (*b*) The sphere does not spin relative to the directed flux. (*c*) There are no radiative heat losses. (*d*) There are no convection losses, either from the surface or in the interior (for a liquid sphere).

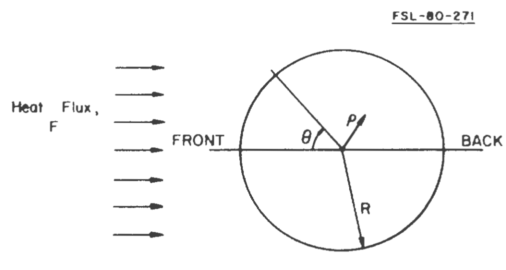

Fig. 1 Geometry

[1] Fusion Studies Laboratory, University of Illinois, Urbana, Ill. 61801.

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