Reprinted from JOURNAL OF HEAT TRANSFER, Vol. 102, No. 1, February 1980

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A Successive Approximation Approach to Problems in Radiative Transfer with a Differential Formulation

The radiation intensity in a gray participating medium is expressed in a differential form. The energy equation for radiative transfer becomes an infinite-order differential equation. Utilizing the method of weighted residuals and introducing some appropriate formulations for the intensity boundary conditions, a method of successive approximations is developed. The solution method is applied to a one-dimensional problem with linear-anisotropic scattering. This problem is chosen because of its practical importance and the availability of exact solutions. A first-order closed-form result, which has never been derived analytically before, is obtained and shown to have good accuracy. Successive higher-order approximate solutions are also presented. These solutions are easily attainable algebraically and converge quickly to the exact result. To illustrate the possible applicability of the solution method for multidimensional problems, the first-order solution to a simple two-dimensional problem is presented. Results show that based on the present approach, reasonably accurate approximate solutions can be generated with some simple mathematical developments.

Introduction

The major difficulty in the analysis of radiative heat transfer in a participating medium lies in the mathematical complexity of the integral transport equation. Except for some simple one-dimensional problems [1, 2], exact solution is extremely difficult to obtain. A great deal of research effort in the past, therefore, had been directed to the development of an effective approximation technique for this complex problem. Milne [3], Eddington [4], Deissler [5], Traugott [6], Cheng [7] and many others have proposed different approximation methods for the radiative transfer problem. Their success, however, is quite limited. While most of them are effective in an accurate heat-flux prediction for the one-dimensional planar problem, they are subject to some common difficulties. First, most of the proposed approximation methods cannot be readily generalized to obtain solutions with better accuracy. High-order approximations based on these methods are either not available or almost impossible to obtain because of the extreme mathematical complexity. Secondly, all of the proposed methods are very difficult, if not impossible, to adopt for problems with multidimensional geometry. A few attempts have been reported [7–10], but they are all either too complicated mathematically or too restrictive in application because of some highly simplifying physical assumptions.

The purpose of this investigation is to reformulate the radiative transfer problem as a boundary-value problem. The governing integral equation is transformed into an infinite-order differential equation. The intensity boundary condition is expressed as an infinite set of linear relations in terms of the medium's temperature and its derivatives at the physical boundary. Utilizing the method of weighted residuals, which, in its various forms, has been used extensively to obtain approximate solutions for many problems in fluid mechanics and heat transfer [11], successive approximate solutions can be readily generated. In contrast to the existing approximation methods, the present solution method has the distinct advantage that solutions with increasing degree of accuracy can be obtained with little difficulty. The general philosophy of the solution technique also appears to be adaptable to problems with multi-dimensional geometry.

To illustrate the effectiveness of the proposed solution method, the problem of radiative heat transfer in a one-dimensional, gray, absorbing, emitting and linear-anisotropic scattering medium will be considered. This problem is chosen mainly because of its practical importance in many engineering problems such as energy transport in porous media and in fire and smoke. Solution to this problem will also demonstrate the superiority of the present method, since none of the existing approximation methods have ever been applied successfully to problems with anisotropic scattering. Based on the method of weighted residuals, successive approximate solutions, converging quickly to the exact result, will be developed. Also obtained is a closed-form solution, which is convenient for practical engineering calculations. To demonstrate the potential effectiveness of the present method for multidimensional problems, the first-order approximation of a simple two-dimensional radiative heat transfer problem with planar geometry is developed. The result is shown to compare favorably with the available exact solution

Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division October 11, 1978.

The One-Dimensional Problem

Physical Model. The physical model chosen for the present analysis is a uniform plane parallel gray medium with two black walls. The medium is assumed to consist of particles which scatter radiative energy anisotropically. For simplicity, the anisotropic scattering property of the medium is approximated by retaining only the first two terms of the Legendre polynomial series expansion for the general phase function (for details, consult [12]). Such scattering is termed linearly anisotropic and has the following mathematical representation

$$p(\cos\theta_0) = 1 + x_0\cos\theta_0 \quad -1 \le x_0 \le 1 \tag{1}$$

where p is the phase function, θ_0 the angle between the incoming and the scattered ray and x_0 a coefficient which indicates physically the amount of anisotropic scattering in the medium.

Governing Equations. With the coordinate system as illustrated in Fig. 1, the equation of transfer can be written as

$$\frac{di}{ds} = -\beta i + a i_b + \frac{\gamma}{4\pi}\int_{\omega'=4\pi} i(\omega')p(\omega,\omega')d\omega' \tag{2}$$

where i denotes the radiation intensity, i_b the blackbody intensity, a the absorption coefficient, γ the scattering coefficient, β the extinction coefficient, s the pathlength and ω the solid angle.

Substituting equation (1) into equation (2) and utilizing the axisymmetric property of the planar geometry, the last term of equation (2) can be integrated over ϕ' from 0 to 2π giving

$$\mu \frac{di}{d\tau_z} + i = (1-\omega_0)i_b + \frac{\omega_0}{2}\int_{-1}^{1} i d\mu + \frac{\omega_0 x_0}{2}\mu \int_{-1}^{1} i\mu d\mu \quad (3)$$

where ω_0 represents the single scattering albedo, τ_z the optical thickness defined by

$$d\tau_z = \beta dz \quad (4)$$

with z being the axis of symmetry and $\mu \equiv \cos\theta$.

Assuming that there is no heat generation, the energy equation for the present one-dimensional system is given simply by

$$\frac{d}{d\tau_z}\int_{-1}^{1} i\mu d\mu = 0. \quad (5)$$

Utilizing equation (5), equation (3) can be integrated over μ from −1 to 1 to yield the following familiar relation

$$i_b = \frac{1}{2}\int_{-1}^{1} i d\mu \quad (6)$$

Substituting equation (6) into equation (3) gives

$$\mu \frac{di}{d\tau_z} = i_b - i + \frac{\omega_0 x_0}{2}\mu \int_{-1}^{1} i\mu d\mu \quad (7)$$

Equation (7) can now be rearranged and differentiated successively to obtain the following set of equations

$$i = i_b - \mu \frac{di}{d\tau_z} + \frac{\omega_0 x_0}{2}\mu \int_{-1}^{1} i\mu d\mu$$
$$\frac{di}{d\tau_z} = \frac{di_b}{d\tau_z} - \mu \frac{d^2 i}{d\tau_z^2} \quad (8)$$
$$\frac{d^n i}{d\tau_z^n} = \frac{d^n i_b}{d\tau_z^n} - \mu \frac{d^{n+1} i}{d\tau_z^{n+1}} \quad (n = 1, 2, 3, \ldots)$$

Note that equation (5) has been used to eliminate all but one of the scattering terms appearing in equations (8).

Combining equations (8) and after some algebraic manipulation, the radiation intensity can be written in the following differential form

$$i = \sum_{k=0}^{\infty} (-1)^k \mu^k \frac{d^k i_b}{d\tau_z^k} - \frac{\omega_0 x_0 \mu}{1 - \frac{\omega_0 x_0}{3}} \sum_{k=1}^{\infty} \left(\frac{1}{2k+1}\right) \frac{d^{2k-1} i_b}{d\tau_z^{2k-1}} \quad (9)$$

Substituting equation (9) into equation (5), the governing equation for the one-dimensional problem based on the present differential formulation becomes

$$\left(\frac{1}{1 - \frac{\omega_0 x_0}{3}}\right) \sum_{k=1}^{\infty} \left(\frac{1}{2k+1}\right) \frac{d^{2k} i_b}{d\tau_z^{2k}} = 0 \quad (10)$$

Equation (10) has the advantage of being a differential equation instead of an integral equation under the traditional formulation. It is, however, of infinite order. In principle, its solution requires an infinite number of boundary conditions.

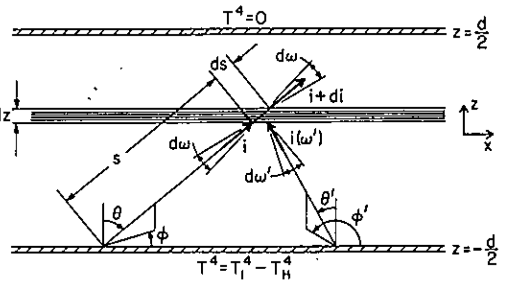

Fig. 1 Coordinate system for radiative transfer in a plane parallel geometry

Boundary Conditions. For the present one-dimensional problem and in fact most problems in radiative heat transfer of practical interest, temperatures of the bounding surfaces are specified. In terms of the radiative intensity, boundary conditions for the present problem are

$$i\left(-\frac{L}{2}, \mu\right) = \frac{\sigma T_1^4}{\pi} \quad 0 \le \mu \le 1 \quad (11)$$

$$i\left(\frac{L}{2}, \mu\right) = \frac{\sigma T_2^4}{\pi} \quad -1 \le \mu \le 0 \quad (12)$$

where σ is the Stefan-Boltzman constant, T_1 the lower wall temperature, T_2 the upper wall temperature and $L = \beta d$ with d representing the distance between the bounding walls. Utilizing equation (9), equations (11) and (12), evaluated at the different values for μ, can be considered as an infinite set of linear relations in terms of the blackbody intensity and its derivatives at the two boundaries. These relations, together with equation (10), thus constitute a complete mathematical description of the present one-dimensional problem.

Approximate Solution. Exact solution to equation (10) and its associated boundary condition is clearly impossible to obtain. But a series of successive approximate solutions, converging to the exact result, can be constructed by the following procedure.

In the Nth approximation, the blackbody intensity in the medium is assumed to be a finite-order power series of the following form:

$$i_b^{(N)} = \sum_{k=0}^{4N-1} A_k \tau_z^k \quad (13)$$

Physically, i_b is expected to be a monotonically increasing smooth function. It also approaches a linear distribution in the limit of $L \to \infty$. Equation (11) should thus be a fairly accurate representation of the blackbody intensity even for moderate values of N.

Equation (13) is obviously not a solution to the governing conservation equation. In fact, when $i_b^{(N)}$ is substituted into equation (10), a residual ϵ_N will be resulted as follows:

$$L_N(i_b^{(N)}) = \epsilon_N \quad (14)$$

where L_N is an operator defined by

Nomenclature

A_j = coefficients defined in equation (13)
a = absorption coefficient
B_0, B_1 = coefficients defined in equation (32)
d = distance between the two walls for the parallel plate system
F_0, F_1 = functions defined by equations (32) and (33)
i = radiation intensity
i_b = blackbody intensity
$i_b^{(N)}$ = Nth approximation of the blackbody intensity
$j = \sqrt{-1}$, equation (23)
$L = \beta d$, optical thickness between the two walls of the parallel plate system
L_N = differential operator in the Nth order approximation, equation (15)
ℓ_x, ℓ_y, ℓ_z = directional cosines
$\underline{\ell}$ = vector with components (ℓ_x, ℓ_y, ℓ_z)
P = phase function
$q^{(1)}$, $q_x^{(1)}$, $q_z^{(1)}$ = first approximation of the heat flux
S = distance along a line of sight
T = temperature
T_1 = temperature of the lower wall
T_2 = temperature of the upper wall
u, v, w = coefficients defined in equations (34) and (35)
W_k = weighting factors for the method of weighted residuals, equation (16)
x_0 = coefficient defined in equation (1)
x, z = spatial coordinate
β = extinction coefficient
γ = scattering coefficient
ϵ_N = residuals in the Nth approximation, equation (14)
θ_0 = scattering angle measured from forward direction to direction of observer, equation (1)
θ = polar angle, Fig. 1
λ = constant defined in equation (23)
$\mu = \cos\theta$
σ = Stefan-Boltzmann constant
τ_x, τ_z = ax, az defined in equation (4)
$\underline{\tau}$ = vector with components (τ_x, τ_y, τ_z)
ω, ω' = solid angle
ω_0 = single scattering albedo

$$L_N = \sum_{k=1}^{2N} \left(\frac{1}{2k+1}\right) \frac{d^{2k}}{d\tau_z^{2k}} \quad (15)$$

To determine the best estimate for the constants A_k's, the present work utilizes the commonly used method of weighted residuals [11]. Multiply equation (14) by a weighting factor W_k and integrate over the one-dimensional space. The Nth order approximation requires that ϵ_N be small in the following sense:

$$\int_{-L/2}^{L/2} W_k L_N(i_b^{(N)}) d\tau_z = 0 \quad (k = 1, 2, \ldots, 2N) \quad (16)$$

The exact form of the weighting factor generally depends on the nature of the considered problem. For the present one-dimensional analysis, since $i_b^{(N)}$ is assumed to be a power series in τ_z, a natural choice for the weighting factor is

$$W_k = \tau_z^{k-1} \quad (17)$$

In the Nth approximation, equation (10) is thus approximated by the following $2N$ algebraic relations

$$\int_{-L/2}^{L/2} \tau_z^{k-1} L_N(i_b^{(N)}) d\tau_z = 0 \quad (k = 1, 2, \ldots, 2N) \quad (18)$$

Physically, it is interesting to note that equation (18) can be interpreted as the conservation of the first $2N$ multi-order moments of the radiation intensity within the medium [13]. Mathematically, equation (18) is identical to requiring that equation (10) is satisfied at $2N$ distinct values of τ_z. In the limit of $N \to \infty$, equations (18) and (10) are clearly equivalent.

To generate the remaining set of equations needed for the determination of A_k's, $2N$ relations must be generated from the boundary conditions. The task of approximating equations (11) and (12) by a finite set of relations is not entirely new. A great number of efforts have been made in this area, because all of the existing approximation techniques require some forms of approximation for the intensity boundary conditions. A survey of the existing works suggests that the following two ideas appear to be the most applicable for the present consideration:

1 *Marshak's Boundary Conditions.* Proposed by Marshak [14], the boundary conditions for the present Nth order approximation are

$$\int_0^1 i\left(\frac{L}{2}, \mu\right) \mu^{2k-1} d\mu = \frac{\sigma T_1^4}{2k\pi} \quad (k = 1, 2, \ldots, N) \quad (19)$$

$$\int_1^0 i\left(\frac{L}{2}, \mu\right) \mu^{2k-1} d\mu = \frac{\sigma T_2^4}{2k\pi}$$

Physically, equations (19) can be interpreted as the conservation of multi-order moments of the radiation intensity across the two boundaries. It is compatible with the physical interpretation of equations (18).

2 *Variational Boundary Conditions.* In his work with the Spherial Harmonics techniques, Pomraning [15] showed that the radiative transfer equation and its associated boundary conditions can be characterized simply by a Lagrangian. Applying the variational principle, the intensity boundary condition can be written in the following form.

$$\int_0^1 d\mu\mu \left[i\left(-\frac{L}{2}, \mu\right) - \frac{\sigma T_1^4}{\pi}\right] \delta i\left(-\frac{L}{2}, -\mu\right) + \int_{-1}^0 d\mu\mu \left[\frac{\sigma T_2^4}{\pi} - i\left(\frac{L}{2}, \mu\right)\right] \delta i\left(\frac{L}{2}, -\mu\right) = 0 \quad (20)$$

where $\delta i(-L/2, -\mu)$ and $\delta i(L/2, -\mu)$ stand for the variation of the trial intensity function at the lower and the upper walls, respectively. In the Nth approximation, if the variation of the blackbody intensity i_b and its first $(N-1)$ derivatives at the two boundaries are assumed to be independent, it can be readily shown that equation (20) is reduced to a set of $2N$ equations.

Equations (18), together with either equations (19) or (20) thus constitute a complete set of equations based on which the coefficient A_k's can be evaluated.

Results and Discussion. For the present one-dimensional problem, calculation shows that both Marshak's boundary conditions and the variational boundary conditions are effective in generating accurate low-order approximate solutions. The first-order approximation using Marshak's boundary conditions, for example, yields the following radiative heat flux expression

$$q^{(1)} = \frac{\sigma T_1^4 - \sigma T_2^4}{1 + \left(\frac{3}{4} - \frac{\omega_0 x_0}{4}\right) L} \quad (21)$$

In the limit of $\omega_0 \to 0$, equation (21) reduces to the familiar diffusion approximation [5]. It is interesting to note that the same expression was developed semi-empirically by Dayan and Tien [16] by matching exact solutions in the optically thick and thin limits. Equation (21) compares very well with the available exact solution [16] for all values of the optical thickness and the scattering albedo.

With the variational boundary condition, the first-order approximation gives a slightly different expression for the radiative heat flux as follows:

$$q^{(1)} = \frac{\sigma T_1^4 - \sigma T_2^4}{\frac{3}{4}(2)^{1/2} + \left(\frac{3}{4} - \frac{\omega_0 x_0}{4}\right) L} \quad (22)$$

While equation (22) is less accurate than equation (21) in the optically thin limit, it is superior in the optically thick limit. When $\omega_0 = 0$, for example, equation (22) agrees quite closely with the optically thick asymptotic heat flux expression obtained from an exact calculation [1, 2].

Results for the radiative heat flux based on the first three approximations and Marshak's boundary conditions, together with the exact result, are presented in Table 1. Some typical results of the temperature profile are shown in Fig. 2. It can be readily observed that the successive approximate solutions converge quickly to the exact solution, independent of the value of the optical thickness and the scattering albedo. It is important to emphasize that for each approximation, the solution is obtained by solving a simple finite-order matrix equation for the constants A_k, a relatively easy task.

The Two-Dimensional Problem

Physical Model. To illustrate the effectiveness of the present method for multidimensional radiative transfer, the present work considers the same parallel-plate system as in the previous one-dimensional problem. For simplicity, the scattering coefficient is now assumed to be zero, and the two-dimensional effect is considered to be generated by the following temperatures at the two boundaries

$$\sigma T_1^4 = e^{j\lambda\tau_x}, \quad T_2 = 0 \quad (23)$$

where $j = \sqrt{-1}$ and λ is a constant which can be interpreted as the inverse wave-length of the sinusoidal temperature variation at the lower wall. In spite of its simplicity, the present problem has important practical applications, because by superposition its solution can be generalized to parallel-plate radiative transfer problems with arbitrary wall temperatures. This problem also serves as a good indication on the effectiveness of the present method because exact solution [17, 18] is available to check the accuracy of the approximate solution.

Governing Equations. Following a similar procedure as in the one-dimensional problem, the radiation intensity for the present two-dimensional system can be written in the following differential form:

$$i(\underline{\tau}, \underline{\ell}) = \sum_{k=0}^{\infty} (-1)^k (\underline{\ell} \cdot \text{grad})^k i_b \quad (24)$$

where

$$\underline{\ell} \cdot \text{grad} = \frac{1}{a}\left[\ell_x \frac{\partial}{\partial x} + \ell_z \frac{\partial}{\partial z}\right]$$

with ℓ_x, ℓ_z being the directional cosines in the x and z direction, respectively. The energy conservation equation becomes

Table 1 Comparison of the first, second and third approximations of the dimensionless heat flux $q_z/(\sigma T_1^4 - \sigma T_2^4)$ with the exact solution for the one-dimensional problem with anisotropic scattering

	$\omega_0 x_0 = -0.7$		$\omega_0 x_0 = 0.0$		$\omega_0 x_0 = 0.7$	
L	First, Second and Third	Exact	First, Second and Third	Exact	First, Second and Third	Exact
0.1	0.915 0.907 0.901	0.901	0.930 0.921 0.916	0.916	0.946 0.937 0.931	0.931
0.5	0.684 0.668 0.662	0.661	0.727 0.709 0.702	0.704	0.777 0.756 0.748	0.750
1.0	0.520 0.507 0.504	0.505	0.571 0.556 0.553	0.553	0.635 0.616 0.612	0.611
3.0	0.265 0.261 0.260	0.260	0.308 0.302 0.302	0.302	0.367 0.359 0.358	0.358
10.0	0.098 0.097 0.097		0.118 0.117 0.117	0.109	0.148 0.147 0.147	

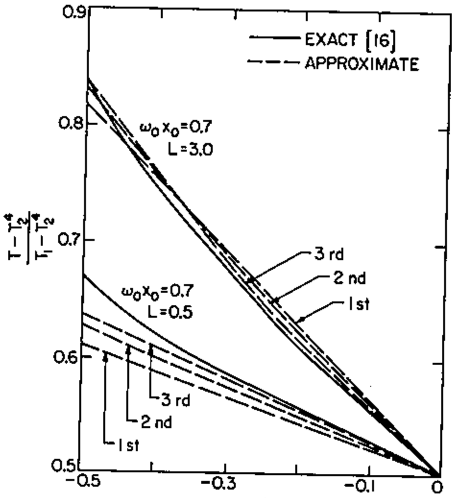

Fig. 2 The comparison of the first, second and third approximation for some typical temperature profile with the exact result [16] for the one-dimensional problem with anisotropic scattering

$$\sum_{k=1}^{\infty} \left(\frac{1}{2k+1}\right) (\text{div} \cdot \text{grad})^k i_b = 0 \quad (25)$$

where div·grad is the two-dimensional Laplacion given by

$$\text{div} \cdot \text{grad} = \frac{1}{a^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right) \quad (25a)$$

Based on equation (25) a series of approximate solutions can again be constructed utilizing the method of weighted residual and imposing the appropriate set of boundary conditions.

Approximate Solution and Discussion. To adopt the solution method presented in the previous sections to the present two-dimensional problem, a number of important modifications are clearly required. The simple power series expansion, for example, does not appear to be an adequate trial solution for the two-dimensional blackbody intensity because, in the limit of $L \to \infty$, i_b should be a function of $\sinh \lambda\tau_z$ and $\cosh \lambda\tau_z$. A different series expansion which possesses the above limiting behavior should be chosen. Correspondingly, a different set of weighting functions W_k should be introduced. The two-dimensional generalization of the operator L_N, as expressed by equation (9) for the one-dimensional case, must be constructed with care. Since the new series expansion for i_b might possess non-vanishing derivatives of all order, some sort of truncation procedure must be introduced in the development of approximate solutions. It can also be shown (as was done for the spherical harmonic method in its application to the neutron transport theory [19]) that without some additional assumptions, the Marshak boundary condition cannot be applied to the multidimensional problem in a mathematically consistent manner. The variational boundary condition is more suitable. It must be emphasized that the applicability of the present solution method to multidimensional problems will remain an open question until all of the above modifications are adequately resolved. A detailed consideration, however, would be quite lengthy and beyond the scope of the present work. Many of these modifications are currently under investigation; and these results will be presented in future publications. For the purpose of demonstrating the potential applicability of the proposed solution method, the present work will consider only a simple first-order approximate solution. As will be demonstrated, the first-order result already gives good agreement with the available exact solution.

Utilizing the symmetry of the present two-dimensional system, the first-order approximation for the blackbody intensity is assumed to be

$$i_b^{(1)} = (A_0 + B_0 e^{\lambda\tau_z} + B_1 e^{-\lambda\tau_z}) e^{j\lambda\tau_x} \quad (26)$$

Similar to equation (8), the governing conservation equation is approximated by

$$\int_{-1/2}^{1/2} L_1(i_b^{(1)}) d\tau_z = 0 \quad (27)$$

where L_1 is a two-dimensional operator defined as

$$L_1 = \frac{1}{a^2}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right] \quad (28)$$

Substitution of equation (26) into equation (27) immediately yields

$$A_0 = 0 \quad (29)$$

The remaining constants B_0 and B_1 must now be determined from the boundary conditions.

Generalizing the development of Pomraning [15], the variational boundary condition for the present two-dimensional problem can be written as

$$\int_0 d\omega \ell_z \left[i\left(\tau_x, -\frac{L}{2}, \ell_{x'} \ell_z\right) - \frac{e^{j\lambda\tau_x}}{\pi}\right] \delta i \left(\tau_{x'} - \frac{L}{2}, -\ell_{x'} - \ell_z\right) + \int_0 d\omega \ell_z \left[-i\left(\tau_{x'} \frac{L}{2'} \ell_{x'} \ell_z\right)\right] \delta i \left(\tau_{x'} \frac{L}{2'} - \ell_{x'} - \ell_z\right) = 0 \quad (30)$$

where $\int_0 d\omega$ and $\int_0 d\omega$ represent integration over the upper and lower hemisphere respectively. Substituting equations (23) and (26) into equation (30), keeping only terms up to the first derivative in τ_z and carrying out all the necessary integration, the following equation is obtained.

$$\left[\frac{1}{16} F_0\left(-\frac{L}{2}\right) + \frac{1}{8} F_1\left(-\frac{L}{2}\right) - \frac{1}{4}\right] \delta F_0\left(-\frac{L}{2}\right) + \frac{1}{64}\lambda^2 F_0\left(-\frac{L}{2}\right) \delta F_0\left(-\frac{L}{2}\right) - \left[\frac{1}{8} F_0\left(-\frac{L}{2}\right) + \frac{9}{32} F_1\left(-\frac{L}{2}\right) - \frac{1}{2}\right] \delta F_1\left(-\frac{L}{2}\right) + \left[\frac{1}{16} F_0\left(\frac{L}{2}\right) - \frac{1}{8} F_1\left(\frac{L}{2}\right)\right] \delta F_0\left(\frac{L}{2}\right)$$

$$+\frac{1}{64}\lambda^2 F_0\left(\frac{L}{2}\right)\delta F_0\left(\frac{L}{2}\right)+\left[\frac{1}{8}F_0\left(\frac{L}{2}\right)-\frac{9}{32}F_1\left(\frac{L}{2}\right)\right]\delta F_1\left(\frac{L}{2}\right)=0 \qquad (31)$$

The functions F_0 and F_1 are introduced in equation (31) for convenience; they are defined as

$$F_0(\tau_z) = 4\pi(B_0 e^{\lambda\tau_z} + B_1 e^{-\lambda\tau_z}) \qquad (32)$$

$$F_1(\tau_z) = \frac{4\pi}{3}\lambda(B_0 e^{\lambda\tau_z} - B_1 e^{-\lambda\tau_z}) \qquad (33)$$

Now, in the use of the variational method one usually assumes that all variations at the boundaries of the system are independent. But the definition of F_0 and F_1 involve only two constants and, since there are just two boundaries in the present problem, variations of F_0 and F_1 at each boundary are clearly not totally independent. For simplicity, the present work assumes that at the two boundaries F_0 and F_1 are linearly related according to

$$F_1\left(\frac{L}{2}\right) - uF_0\left(\frac{L}{2}\right) = 0 \qquad (34)$$

$$vF_1\left(-\frac{L}{2}\right) + wF_0\left(-\frac{L}{2}\right) = 1 \qquad (35)$$

where u, v and w are constant and, from a simply physical consideration, can be shown to be positive. Using equations (34) and (35) in equation (31) to eliminate all the F_1 terms and setting the coefficient of $F_0(L/2)\delta F_0(L/2)$, $\delta F_0(-L/2)$ and $F_0(-L/2)\delta F_0(-L/2)$ to be zero yields the following algebraic equations for u, v and w.

$$\left(\frac{1}{16}+\frac{\lambda^2}{64}\right) - \frac{9}{32}u^2 = 0 \qquad (36)$$

$$\left(\frac{1}{16}+\frac{\lambda^2}{64}\right) - \frac{9}{32}\left(\frac{w}{v}\right)^2 = 0 \qquad (37)$$

$$\frac{1}{8v} - \frac{1}{4} + \frac{w}{v}\left(\frac{9}{32v} - \frac{1}{2}\right) = 0 \qquad (38)$$

The solution of equations (36) to (38) is

$$u = \left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2} \qquad (39)$$

$$v = \frac{1+\frac{9}{4}\left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2}}{2+4\left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2}} \qquad (40)$$

$$w = \left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2}\left[\frac{1+\frac{9}{4}\left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2}}{2+4\left(\frac{2}{9}+\frac{\lambda^2}{18}\right)^{1/2}}\right] \qquad (41)$$

Equations (34) and (35) can now be solved to yield

$$B_0 = \frac{\frac{1}{4\pi}\left(u-\frac{\lambda}{3}\right)e^{-\lambda(L/2)}}{v\left[\left(u-\frac{\lambda}{3}\right)^2 e^{-\lambda L} - \left(u+\frac{\lambda}{3}\right)^2 e^{\lambda L}\right]} \qquad (42)$$

$$B_1 = \frac{-\frac{1}{4\pi}\left(u+\frac{\lambda}{3}\right)e^{\lambda(L/2)}}{v\left[\left(u-\frac{\lambda}{3}\right)^2 e^{-\lambda L} - \left(u+\frac{\lambda}{3}\right)^2 e^{\lambda L}\right]} \qquad (43)$$

Physically, the interesting quantities are the temperature and the radiative heat flux. In terms of the constants B_0 and B_1, they are

$$i_b^{(1)} = [B_0 e^{\lambda\tau_z} + B_1 e^{-\lambda\tau_z}]e^{j\lambda\tau_x} \qquad (44)$$

$$q_x^{(1)} = -\frac{4\pi}{3}j\lambda[B_0 e^{\lambda\tau_z} + B_1 e^{-\lambda\tau_z}]e^{j\lambda\tau_x} \qquad (45)$$

$$q_z^{(1)} = -\frac{4\pi}{3}\lambda[B_0 e^{\lambda\tau_z} - B_1 e^{-\lambda\tau_z}]e^{j\lambda\tau_x} \qquad (46)$$

In a recent work, Breig and Crosbie [17, 18] obtained the exact solution for the radiative heat flux and the gas temperature at the two bounding walls for the present problem. Comparisons between equation (46) and the exact result at the upper and lower walls are shown in Figs. 3 and 4, respectively. The agreement is quite satisfactory. Even though exact solutions for q_z and q_x at arbitrary values of τ_z and τ_x are not available, it is reasonable to expect that they should be of the same order of accuracy as at the two boundaries. The temperature profile prediction, however, is less satisfactory. It can be shown that equation (44), when evaluated at the lower wall, yields an incorrect asymptotic value in the limit of $\lambda \rightarrow \infty$. The accuracy improves at the region near the upper wall. A comparison between equation (44) and the exact solution for the gas temperature at the upper wall, for example, is shown in Fig. 5.

To further demonstrate the relative accuracy and efficiency of the present solution method compared with the existing approximation techniques, the two-dimensional problem with a constant temperature at the upper wall and a step temperature at the lower wall is now considered based on the above first-order result. This problem is chosen largely because an exact solution generated by the Monte Carlo technique [20] and a fairly accurate approximate solution generated

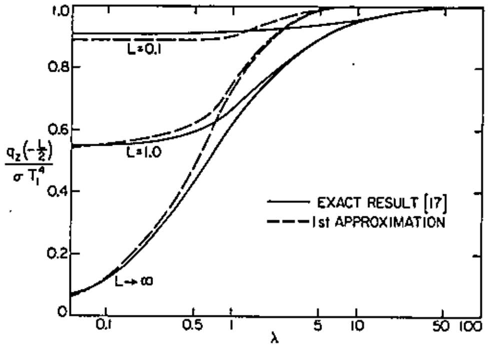

Fig. 3 Comparison between the first approximation and the exact result of the radiative heat flux at the lower boundary for the two-dimensional problem

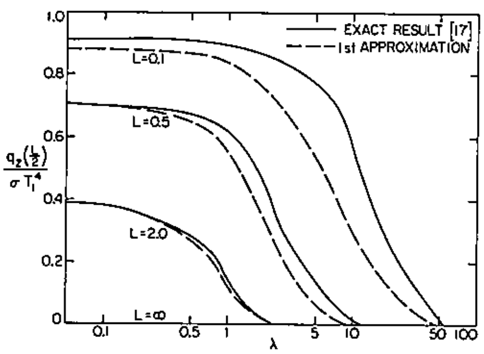

Fig. 4 Comparison between the first approximation and the exact result of the radiative heat flux at the upper wall of the two-dimensional problem

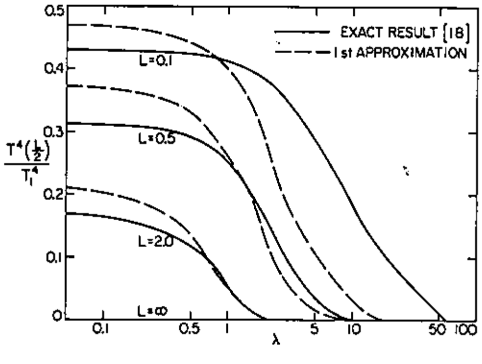

Fig. 5 Comparison between the first approximation and the exact result of the dimensionless gas temperature at the upper boundary for the two-dimensional problem

by a modified differential method [21] are both available in the literature. They can thus serve conveniently as a basis for a direct comparison.

Specifically, the boundary conditions for this sample two-dimensional problem are assumed to be the following normalized forms:

$$\sigma T_1^4 = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \quad (47a)$$

$$\sigma T_2^4 = 0 \quad (47b)$$

By a Fourier series expansion, it can be readily shown that equation (47a) can be written as

$$\sigma T_1^4 = \lim_{L_x \to \infty} \frac{4}{\pi} \sum_{n=0}^{\infty} (2n+1) \sin \lambda_n \tau_x \quad (48)$$

with

$$\lambda_n = \frac{(2n+1)\pi}{L_x} \quad (49)$$

Utilizing equation (46) and a simple superposition, the first-order approximations for the radiative heat flux at the upper and lower walls are obtained. They are

$$q_z\left(\tau_{x'} - \frac{L}{2}\right) = \lim_{L_x \to \infty} \sum_{n=0}^{\infty} \left(-\frac{16}{3}\lambda_n\right) \times [B_0(\lambda_n)e^{-\lambda_n(L/2)} - B_1(\lambda_n)e^{\lambda_n(L/2)}] \sin \lambda_n \tau_x \quad (50)$$

$$q_z\left(\tau_x, \frac{L}{2}\right) = \lim_{L_x \to \infty} \sum_{n=0}^{\infty} \left(-\frac{16}{3}\lambda_n\right) \times [B_0(\lambda_n)e^{\lambda_n(L/2)} - B_1(\lambda_n)e^{-\lambda_n(L/2)}] \sin \lambda_n \tau_x \quad (51)$$

Physically, the interesting heat flux results are those near the point $\tau_x = 0$. For $\tau_x/L \leq 5.0$, calculations show that the $L_x \to \infty$ results are practically indistinguishable from the results with $L_x/L = 100$. Assuming $L_x/L = 100$, the heat flux at various locations of the upper and lower walls with $L = 0.1, 0.5, 1.0, 2.0$ and 5.0 are tabulated and presented in Tables 2 and 3. By some additional superposition with the one-dimensional result, the corresponding heat flux values for the problem with the unnormalized boundary conditions considered in [20] and [21] can be easily generated. The agreement is excellent for all cases. In a direct comparison with the modified differential method [21], results in Tables 2 and 3 have the same degree of accuracy as those presented graphically in [21]. But the present first-order approximation represents a substantial reduction in mathematical complexities. Equation (46) can also be applied to generate heat flux results for planar systems with arbitrary wall temperature with relatively little effort.

Table 2 The heat flux at different locations of the lower wall for the sample two-dimensional problem with boundary conditions as described by equation (47)

τ_x/L / L	0	1.0	2.0	3.0	4.0	5.0	∞
0.1	∞	1.000	0.966	0.942	0.925	0.914	0.880
0.5	∞	0.869	0.771	0.727	0.708	0.700	0.697
1.0	∞	0.700	0.591	0.561	0.553	0.551	0.551
2.0	∞	0.483	0.403	0.391	0.390	0.390	0.390
5.0	∞	0.241	0.209	0.207	0.207	0.207	0.207

Table 3 The heat flux at different locations of the upper wall for the sample two-dimensional problem with boundary conditions as described by equation (47)

τ_x/L / L	0	1.0	2.0	3.0	4.0	5.0	∞
0.1	0	0.452	0.644	0.731	0.779	0.808	0.880
0.5	0	0.436	0.600	0.658	0.681	0.690	0.697
1.0	0	0.338	0.510	0.541	0.549	0.551	0.551
2.0	0	0.303	0.376	0.388	0.390	0.390	0.390
5.0	0	0.177	0.205	0.207	0.207	0.207	0.207

Concluding Remarks

A successive approximation approach is proposed for the solution of problems in radiative transfer based on a differential formulation. The problem is applied successfully to a one-dimensional problem including the effect of anisotropic scattering. Both first-order closed-form results and higher-order approximate solutions are presented. These solutions are easily attainable algebraically and converge quickly to the exact result.

Application of the present solution method to multidimensional problems encounters some difficulties. Important questions concerning the proper selection of trial solution, the approximation of the intensity boundary condition and the development of a consistent truncation criteria for the multi-dimensional governing equation are still unresolved and require further investigation. Nevertheless, a direct generalization of the one-dimensional first-order approximation method to a two-dimensional problem with planar geometry is demonstrated to be successful. The heat flux prediction agrees well with the available exact results and results generated by other approximation techniques. The closed-form solution is mathematically simple and can be used to generate heat transfer results for problems with arbitrary wall temperature distribution.

References

1 Heaslet, M. A. and Warming, R. F., "Radiative Trasnport and Wall Temperature Slip in an Absorbing Planar Medium," *International Journal Heat Mass Transfer*, Vol. 8, 1965, pp. 979–994.

2 Heaslet, M. A. and Warming, R. F., "Radiative Transportant in An Absorbing Medium II. Prediction of Radiative Source Function," *International Journal Heat Mass Transfer*, Vol. 10, 1967, pp. 1413–1427.

3 Milne, E. A., "Thermodynamics of Stars," *Handbuch der Astrophysik*, Vol. 3, Spring-Verlag, Berlin, 1930, pp. 65–135.

4 Eddington, A. S., *The Internal Constitution of Stars*, Dover Publication, Inc., New York, 1959.

5 Deissler, R. G., "Diffusion Approximation for Thermal Radiation in Gases with Jump Boundary Conditions," ASME JOURNAL OF HEAT TRANSFER, Vol. 86, 1964, pp. 240–246.

6 Traugott, S. C., "A Differential Approximation For Radiative Transfer with Application to Normal Shock Structure," *Proc. 1963 Heat Transfer Fluid Mech. Inst.*, 1963, pp. 1–13.

7 Cheng, P., "Two-dimensional Radiating Gas Flow by a Moment Method," *AIAA Journal*, Vol. 2, 1964, pp. 1662–1664.

8 Olfe, D. B., "A Modification of the Differential Approximation for Radiative Transfer," *AIAA Journal*, Vol. 5, 1967, pp. 638–643.

9 Taitel, Y., "Formulation of Two-Dimensional Radiant Heat Flux for Absorbing—Emitting Plane Layer with Non-isothermal Bounding Walls," *AIAA Journal*, Vol. 7, 1969, pp. 1832–1837.

10 Pomraning, G. C., *The Equation of Radiation Hydrodynamics*, Pergamon Press, New York, 1973.

11 Finlayson, B. A., *The Method of Weighted Residuals and Variational Principles*, Academic Press, New York, 1972.

12 Hottel, H. C. and Sarofim, A. F., *Radiative Transfer*, McGraw-Hill, New York, 1967.

13 Yuen, W. W., "A Differential Formulation of Radiative Transfer and Its Application to One-Dimensional and Two-Dimensional System," Ph.D. Dissertation, University of California, Berkeley, May 1977.

14 Marshak, R. E., "Note on the Spherial Harmonic Method as Applied to the Milne Problem for a Sphere," *Phys. Rev.*, Vol. 71, 1947, p. 443.

15 Pomraning, G. C., "Variational Boundary Conditions for the Spherical Harmonics Approximation to the Neutron Transport Equation," *Annals of Physics*, Vol. 27, 1964, pp. 193–215.

16 Dayan, A. and Tien, C. L., "Heat Transfer in Gray Planar Medium with Linear Anisotropic Scattering," ASME JOURNAL OF HEAT TRANSFER, Vol. 97, 1975, pp. 391–396.

17 Breig, W. F. and Crosbie, A. L., "Two-Dimensional Radiative Equilibrium: Boundary Emissive Powers for a Finite Medium Subjected to Cosine Varying Radiation," *Journal of Quantitative Spectroscopy Radiative Transfer*, Vol. 14, 1974, pp. 1209–1237.

18 Breig, W. F. and Crosbie, A. L., "Two-Dimensional Radiative Equilibrium: Boundary Fluxes for A Finite Medium Subjected to Cosine Varying Radiation," *Journal of Quantitative Spectroscopy Radiative Transfer*, Vol. 15, 1975, pp. 163–179.

19 Davidson, B. and Sykes, J. B., *Neutron Transport Theory*, The Claredon Press, Oxford, 1957.

20 Murakami, M., "Direct Monte Carlo Simulation of Two-Dimensional Radiative Heat Transfer in Absorbing-Emitting Medium Bounded by the Non-Isothermal Gray Walls," *Proceedings of the Ninth International Symposium on Space Technology and Science*, Agne Publ. 1971, pp. 407–416.

21 Modest, M. F., "Two-Dimensional Radiative Equilibrium of a Gray Medium in a Plane Layer Bounded by Gray Nonisothermal Walls," ASME JOURNAL OF HEAT TRANSFER, Vol. 96, 1974, pp. 483–488.