heat transfer between parallel plates with top and bottom cooling and correlated their analytical predictions in the range $0.1 < Pr < 10^4$ and $10^4 < Re < 10^6$ by

$$Nu = 12 + 0.03\, Re^{a_1} Pr^{a_2} \quad (4)$$

where

$$a_1 = 0.88 - \frac{0.24}{(3.6 + Pr)} \quad (5)$$

$$a_2 = 0.33 + 0.5e^{-0.6Pr} \quad (6)$$

For the Prandtl numbers of the present data set, the second term in equation (6) is negligible compared with the first. In equation (5), the second term is approximately −0.015 for the Prandtl numbers of the present data set, so that the Shibani-Ozisik formula becomes

$$Nu = 12 + 0.03\, Re^{0.865} Pr^{0.33} \quad (7)$$

Equation (7) is also presented in Fig. 3 and is seen to overpredict the Nusselt number by about 50 percent.

Several reasons are offered for the differences between the data and the predictions. First, it is difficult to achieve the idealized flow conditions implicit in the analyses and there may have been a high turbulence level in the recirculating flow. We also suspect the steel plate had some roughness in the present experiments, and in the Reynolds range of these experiments, this could well have contributed to a higher Nusselt number. Second, the aspect ratio B/D_h varied from 1.5 for the Ashton-Hsu data sets to 5.5 for the present study, while the Petukhov-Popov formula is applicable to a round cross section roughly equivalent to a $B/D_h = 1$ and the Shibani-Ozisik formula is applicable to a $B/D_h = \infty$.

Conclusions

The main objectives of the present experiments were to obtain new data on the effect of aspect ratio B/D_h on heat transfer and compare the data to existing theoretical and empirical results. Existing data for round cross sections roughly equivalent to $B/D_h = 1$ are represented by various empirical formulae such as the Petukhov-Popov formula. This formula underpredicts the results of Ashton-Hsu for $B/D_h = 1.5$ and for the present data for $B/D_h = 5.5$. The Shibani-Ozisik formula, applicable to a $B/D_h = \infty$ overpredicts the heat transfer rates. However, the present data and the Ashton-Hsu data are inadequate to provide a relationship for the heat transfer rates as a function of B/D_h.

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A Simplified Approach to Shape-Factor Calculation between Three-Dimensional Planar Objects

W. W. Yuen[1]

Introduction

The objective of the present work is to show that by some simple algebraic manipulation of known shape factors, the shape factor between a differential plane area element and an arbitrary three-dimensional planar object with straight edges can be written in closed-form analytically. The shape factor between two planar objects in which one of them has straight edges is reduced to a simple integration of a positive-definite nonsingular function over only one surface area. Compared to the traditional method of direct numerical integration over both surface areas, the present method represents a substantial reduction of the computational effort. A detailed example calculation is presented to demonstrate the utility of the present approach.

Analysis

The basic idea of the present approach is to express the shape factor between a differential plane area element and a three-dimensional object with straight edges in terms of known shape factors. The expression fundamental to the present analysis is the shape factor between a differential plane area element dA_0 and a right triangle A_1 in a parallel plane. Consider the two planes with the orientation and geometry as shown in Fig. 1. The shape factor between dA_0 and A_1 can be obtained from a simple contour integration [1] as

$$F_{dA_0-A_1} = \frac{u}{2\pi(u^2+d^2)^{1/2}} \tan^{-1}\frac{v}{(u^2+d^2)^{1/2}} \quad (1)$$

Equation (1) can now be utilized to calculate the shape factor between dA_0 and an arbitrary triangle A_1 with orientation as shown in Fig. 2. Using superposition, it can be readily shown that the shape factor is given by

$$F_{dA_0-\overline{A_1}} = F_{dA_0-\overline{014}} + F_{dA_0-065} - F_{dA_0-012} - F_{dA_0-023} - F_{dA_0-063} \quad (2)$$

where F_{dA_0-ijk} stands for the shape factor between dA_0 and the right triangle with vertices i, j, and k. Utilizing equation (1), and after some algebraic manipulation, equation (2) becomes

$$F_{dA_0-A} = G(x_0, x_1, x_2, y_2, d) \quad (3)$$

with $G(x_0, x_1, x_2, y_2, d)$

$$= \frac{x_1 y_2}{2\pi[(x_1^2+d^2)y_2^2 + d^2(x_1-x_2)^2]^{1/2}} \times \tan^{-1}\frac{[(x_1^2+d^2)y_2^2+d^2(x_1-x_2)^2]^{1/2}}{[d^2+x_1x_2]}$$
$$- \frac{x_0 y_2}{2\pi[(x_0^2+d^2)y_2^2 + d^2(x_0-x_2)^2]^{1/2}} \times \tan^{-1}\frac{[(x_0^2+d^2)y_2^2+d^2(x_0-x_2)^2]^{1/2}}{[d^2+x_0x_2]} \quad (3a)$$

It is interesting to note that equation (3) is applicable for both cases with $0 \le x_0 \le x_1 \le x_2$ or $0 \le x_0 \le x_2 \le x_1$.

Equation (3) is developed basically for cases in which A_1 is a triangle with one of its edges lying on the x axis or the y axis. But with the aid of some additional shape-factor algebraic manipulation, equation (3) is sufficient to be applied for other possibilities. Consider the two triangles with arbitrary dimensions and locations as shown in Figs.

[1] Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, Santa Barbara, CA 93106.

Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division July 30, 1979.

3(a) and 3(b). In Fig. 3(a) it can be readily observed that after subdividing the triangle into seven smaller triangles, each of the smaller triangles is of the same general orientation as that of Fig. 2. Similarly, after a simple rotation of the coordinate system in Fig. 3(b), the two smaller triangles as shown have one of their axes lying on the x axis. Equation (3) can be applied to each of these smaller triangles. The shape factor between dA_0 and any parallel triangle can thus be written as a sum of finite numbers of terms, each of which is an expression similar to equation (3). This argument can be further extended to yield the shape factor between dA_0 and a parallel polygon. It is a well known fact in geometry that any n-sided polygon can be composed into n-2 triangles. The shape factor between dA_0 and an n-sided polygon is therefore a sum of n-2 terms, each of which is just the shape factor between dA_0 and a parallel triangle.

Finally, the shape factor between dA_0 and a polygon which is not parallel to dA_0 can also be calculated based on the present analysis. The polygon must first be projected along its different lines of sight from dA_0 to its various vertices onto a plane parallel to dA_0. The shape factor between dA_0 and an arbitrary polygon is simply equal to the shape factor between dA_0 and the line-of-sight projection of that polygon. The previous analysis can thus be applied.

For any three-dimensional planar object with straight edges, the shape factor between dA_0 and the object's different faces can now be written, based on the present approach, as a finite sum of expressions similar to equation (3). Taking dA_0 to be a differential area element on a second object, the shape factor between two finite objects can be obtained by a simple integration. It is important to note that the present approach requires integration over only one surface area. The integral, which consists of a finite number of terms similar to equation (3), is positive-definite and without any undesirable singularity. In contrast with a direct numerical integration over the general shape-factor expression, this represents a substantial reduction in computational effort. It must also be emphasized that the restriction that one of the two objects must be planar and have straight edges is really not too severe since any curve surfaces can be approximated to an arbitrary degree of accuracy by a sum of planar surfaces.

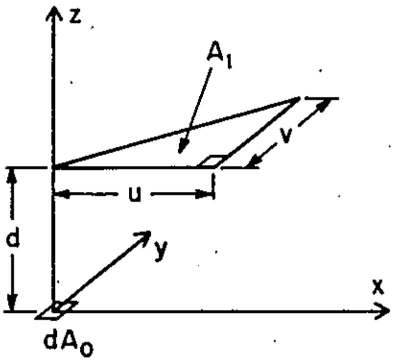

Fig. 1 Coordinate system for the calculation of shape factor between a differential plane-area element and a parallel right triangle

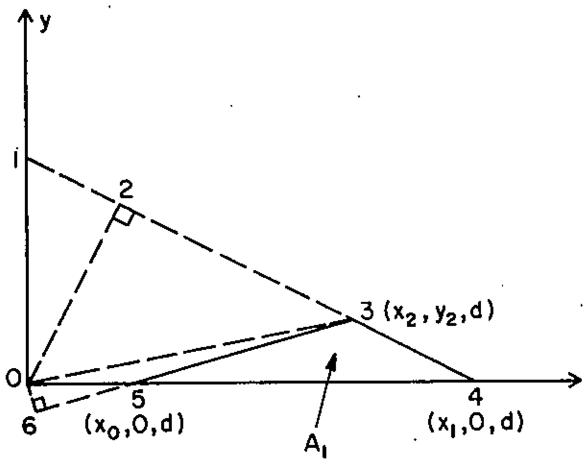

Fig. 2 The geometry and orientation of A_1 relative to the $x - y$ axis for equation (2)

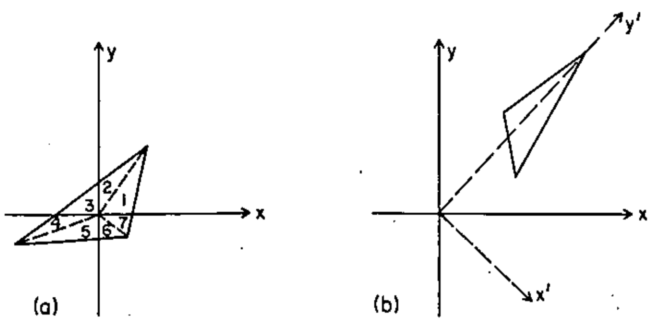

Fig. 3 Examples of how to subdivide an arbitrary triangle into smaller triangles with the general orientation and geometry as in Fig. 2(d)

Examples of Application

To illustrate the utility of the present approach for actual shape-factor evaluation, an example is now presented.

This example deals with the shape factor between two rectangular plates of identical dimension $a \times b$. The two plates are separated by a distance h and one of the plates is oriented at an angle β with respect to the other. The detailed coordinate system is shown in Fig. 4.

The system is clearly symmetric with respect to the x axis. It suffices, therefore, to calculate $F_{dA_I - A_{II}}$ only for values of $x > 0$ and $y > 0$. Consider now a differential area element dA_I at (x, y, o), and the line-of-sight projection of A_{II} onto the plane $z = h + b \sin \beta$ can be

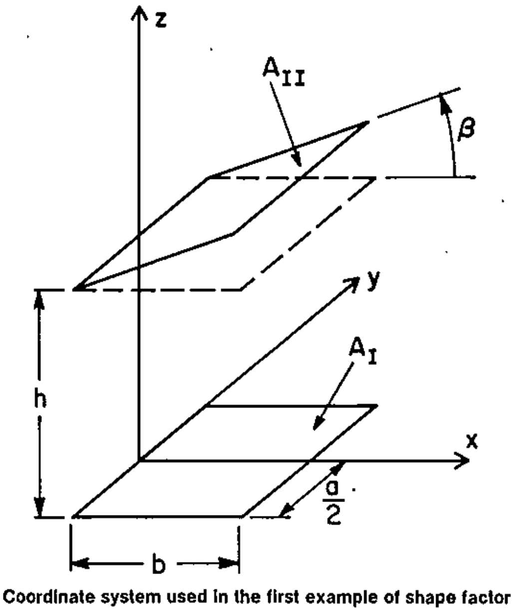

Fig. 4 Coordinate system used in the first example of shape factor calculation

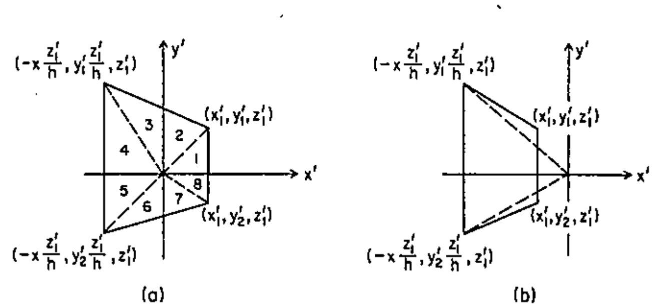

Fig. 5 The line-of-sight projection of A_{II} from dA_I onto the plane $z = h + b \sin \beta$. The coordinates (x', y', z') are measured from dA_I. $(x_1' = b \cos \beta - x$, $y_1 = a/2 - y$, $y_2' = a/2 - y$, $z_1 = h + b \sin \beta)$

readily generated. Depending on the value of x, the relative location of the projected area with respect to (x, y, o) has two possibilities as illustrated in Fig. 5. The shape factor between dA_I and A_{II} can now be readily obtained by direct application of equation (3). For $x < b\cos\beta$, the shape factor is given by

$$F_{dA_I-A_{II}} = \sum_{i=1}^{8} F_{dA_I-A_i} \quad (4)$$

where the A_i's are the smaller areas as denoted in Fig. 5(a) with

$$F_{dA_I-A_1} = G\left(0, b\cos\beta - x, b\cos\beta - x, \frac{a}{2} - y, h + b\sin\beta\right) \quad (5)$$

$$F_{dA_I-A_2} = G\left(0, m, \frac{a}{2} - y, b\cos\beta - x, h + b\sin\beta\right) \quad (6)$$

$$F_{dA_I-A_3} = G\left(0, m, \left(\frac{a}{2} - y\right)\left(1 + \frac{b}{h}\sin\beta\right), x\left(1 + \frac{b}{h}\sin\beta\right), h + b\sin\beta\right) \quad (7)$$

$$F_{dA_I-A_4} = G\left(0, x, x, \frac{a}{2} - y, h\right) \quad (8)$$

$$F_{dA_I-A_5} = G\left(0, x, x, \frac{a}{2} + y, h\right) \quad (9)$$

$$F_{dA_I-A_6} = G\left(0, n, \left(\frac{a}{2} + y\right)\left(1 + \frac{b}{h}\sin\beta\right), x\left(1 + \frac{b}{h}\sin\beta\right), h + b\sin\beta\right) \quad (10)$$

$$F_{dA_I-A_7} = G\left(0, n, \frac{a}{2} + y, b\cos\beta - x, h + b\sin\beta\right) \quad (11)$$

$$F_{dA_I-A_8} = G\left(0, b\cos\beta - x, b\cos\beta - x, \frac{a}{2} + y, h + b\sin\beta\right) \quad (12)$$

and

$$m = \frac{\left(\frac{a}{2} - y\right)(h + b\sin\beta)\cos\beta}{h\cos\beta + x\sin\beta} \quad (13)$$

$$n = \frac{\left(\frac{a}{2} + y\right)(h + b\sin\beta)\cos\beta}{h\cos\beta + x\sin\beta} \quad (14)$$

For $x > b\cos\beta$, the shape factor is

$$F_{dA_I-A_{II}} = \sum_{i=1}^{4} F_{dA_I-A_i} \quad (15)$$

where the A_i's are areas as denoted in Fig. 5(b) and

$$F_{dA_I-A_1} = G\left(0, x, x, \frac{a}{2} - y, h\right) - G\left(0, p, p, \frac{p}{x}\left(\frac{a}{2} - y\right), h + b\sin\beta\right) \quad (16)$$

$$F_{dA_I-A_2} = G\left(0, x, x, \frac{a}{2} + y, h\right) - G\left(0, p, p, \frac{p}{x}\left(\frac{a}{2} + y\right), h + b\sin\beta\right) \quad (17)$$

$$F_{dA_I-A_3} = G\left(\frac{pq}{x}, \left(1 + \frac{b\sin\beta}{h}\right)q, q - \frac{xh}{q}\cos\beta, \left(\frac{a}{2} - y\right)\frac{b}{q}\cos\beta, h + b\sin\beta\right) \quad (18)$$

$$F_{dA_I-A_4} = G\left(\frac{pr}{x}, \left(1 + \frac{b\sin\beta}{h}\right)r, r - \frac{xh}{r}\cos\beta, \left(\frac{a}{2} + y\right)\frac{b}{r}\cos\beta, h + b\sin\beta\right) \quad (19)$$

with

$$p = x - b\cos\beta \quad (20)$$

$$q = \left(x^2 + \left(\frac{a}{2} - y\right)^2\right)^{1/2} \quad (21)$$

$$r = \left(x^2 + \left(\frac{a}{2} + y\right)^2\right)^{1/2} \quad (22)$$

By a simple application of the reciprocity relation, the shape factor between A_I and A_{II} is simply given by

$$F_{A_{II}-A_I} = 2\int_0^{a/2}\int_0^{b} F_{dA_I-A_{II}}\,dxdy \quad (23)$$

Equation (23) can be readily evaluated numerically. Results for various values of a, b, h, and β are presented in Table 1. As expected, the shape factor decreases with increasing h and increasing β. For the case with $\beta = 0$ and the case with $\beta = \pi/2$ and $h = 0$, the present solution gives exact agreement with the available results [1].

Table 1 Shape factors between two rectangular plates with different aspect ratio (b/a), β, and a/h.

β	b/a / a/h	0.2	0.5	1.0	2.0	5.0
	0.5	0.001	0.003	0.009	0.018	0.024
$\pi/2$	1.0	0.004	0.016	0.033	0.045	0.041
	2.0	0.017	0.052	0.076	0.078	0.057
	∞	0.271	0.240	0.200	0.149	0.087
	0.5	0.010	0.023	0.040	0.060	0.071
$\pi/4$	1.0	0.032	0.070	0.108	0.132	0.119
	2.0	0.086	0.165	0.213	0.216	0.157
	∞	0.588	0.547	0.479	0.388	0.234
	0.5	0.015	0.036	0.069	0.117	0.178
0	1.0	0.049	0.117	0.200	0.286	0.359
	2.0	0.135	0.286	0.415	0.509	0.573
	∞	1.000	1.000	1.000	1.000	1.000

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Heat Transfer through Irradiated, Semi-transparent Layers at High Temperature

R. Viskanta[1] and D. M. Kim[1]

Nomenclature

D_n = function defined as $\int_0^1 \mu^{n-2}\exp(-t/\mu)\beta(\mu)d\mu$
$E_{b\lambda}$ = Planck's black body distribution function
E_n = exponential integral defined as $\int_0^1 \mu^{n-2}\exp(-t/\mu)d\mu$
$\mathcal{F}$ = radiative flux in the y-direction
F_i = external flux incident on gas-material interface
k = thermal conductivity of material
n = index of refraction
q = total (conductive + radiative) heat flux in y-direction
R_n = function defined as $\int_0^1 r_i(\mu)\mu^{n-2}\exp(-t/\mu)\beta(\mu)d\mu$
r_i = reflectivity of gas-material interface
T = temperature
t_i = transmissivity of gas-material interface
T_n = function defined as $\int_0^1 t_i(\mu')(\mu')^{n-2}\exp(-t/\mu')\beta(\mu)d\mu'$
y = distance measured from the opaque wall

[1] Heat Transfer Laboratory, School of Mechanical Engineering, Purdue University, West Lafayette, Ind. 47907.

Contributed by the Heat Transfer Division for publication in The JOURNAL OF HEAT TRANSFER. Manuscript received by The Heat Transfer Division June 21, 1979.