Application of the P-1 Approximation to Radiative Heat Transfer in a Nongray Medium

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Nomenclature

a_w = absorption coefficient at wavenumber w
A_p = Planck's mean absorption coefficient
$A_{iw} (i = 0, 1, 2, 3)$ = expansion coefficients defined by equation (5)
d = spectral line spacing
e_{bw} = blackbody emissive power
e_{bw1} = blackbody emissive power evaluated at T_1
e_{bw2} = blackbody emissive power evaluated at T_2
e_{bw0} = blackbody emissive power evaluated at $z = 0$
$e_{bw0}^{(1)} = \partial e_{bw}/\partial z \ (z = 0)$
e_{b1i} = blackbody emissive power evaluated at w_i and T_1
e_{b2i} = blackbody emissive power evaluated at w_i and T_2
e_{b0i} = blackbody emissive power evaluated at w_i and $z = 0$
$e_{b0i}^{(1)} = \partial e_{bw}/\partial z \ (z = 0, w = w_i)$
$F_i^{(k)}$ = integral of F_{kw} over the ith band
F_{kw} = function defined by equation (8)
$G_i^{(k)}$ = integral of G_{kw} over the ith band
G_{kw} = function defined by equation (9)
i_w = radiative intensity
$i_{bw} = e_{bw}/\pi$
$i_{bw1} = e_{bw1}/\pi$
$i_{bw2} = e_{bw2}/\pi$
L = separation between two plates for the one-dimensional system
$L_w = a_w L$
q = radiative heat flux
Q = heat generation rate
S = mean line intensity
T_1 = temperature of the lower boundary
T_2 = temperature of the upper boundary
z = coordinate
γ = line half-width
$\eta = z/L$
$\mu = \cos\theta$
w = wavenumber
ρ = density of gas
$\psi_{n,w}$ = spherical harmonic n-moment

1 Introduction

Interest in predicting the heat transfer rate through an absorbing emitting medium has been increasing as more problems with high temperature have arisen. But realistic solutions to these problems, even in a very simple geometric system, are difficult to obtain because of the frequency and temperature-dependent radiation properties of the medium.

In the present work the P-1 approximation method, which has been demonstrated [1] to be effective in generating accurate approximate solutions to the gray problem, will be generalized for nongray problems. In contrast to most of the existing approaches [2–4], the present method has the advantage that all solutions will be formulated in terms of the spectral absorption coefficient. The method is thus applicable for all media including those for which correlations of total band absorptance are not available. For gases, the present technique will be demonstrated to be superior. Even for situations in which the simultaneous effects of all absorption bands are considered, only simple iterations are required for its solution.

2 Mathematical Formulation

The mathematical development for the P-1 approximation is well known. In essence, the transfer equation and the energy equation are combined to yield

$$\frac{\partial \psi_{1,w}}{\partial z} + a_w \psi_{0,w} = 4 a_w e_{bw} \tag{1}$$

$$\frac{\partial \psi_{0,w}}{\partial z} + 3 a_w \psi_{1,w} = 0 \tag{2}$$

where $\psi_{0,w}$ and $\psi_{1,w}$ are the zeroth and first moment of the spectral intensity function at wavenumber w, a_w the absorption coefficient, e_{bw} the blackbody emission power, and z the coordinate. The energy equation becomes

$$\frac{d}{dz} \int_0^\infty \psi_{1,w} dw = Q \tag{3}$$

with Q being the internal heat generation rate. As in the gray analysis, the intensity boundary conditions are replaced by Marshak's boundary condition. Only two of such conditions are required for the P-1 approximation. They are

$$\int_0^1 i_w \mu d\mu = \frac{1}{2} i_{bw1} \quad \text{at } z = -\frac{L}{2}$$

$$\int_{-1}^0 i_w \mu d\mu = \frac{-1}{2} i_{bw2} \quad \text{at } z = \frac{L}{2} \tag{4}$$

where $\mu = \cos\theta$, $i_{bw1} = i_{bw}(T_1)$, $i_{bw2} = i_{bw}(T_2)$. In the above equations, the two plates are assumed to be at temperature T_1 and T_2 and located at $z = -L/2$ and $z = L/2$, respectively.

Despite their apparent simplicity, equations (1–4) are still difficult to solve exactly. A simple approximate solution, however, can be generated by assuming that the average intensity function, $\psi_{0,w}$ is given by

$$\psi_{0,w} = A_{0w} + A_{1w} z + A_{2w} z^2 + A_{3w} z^3 \tag{5}$$

Substituting equation (5) into equation (2), a similar polynomial expression for ψ_{1w} can be generated. Utilizing equation (4) and requiring that coefficients for the zeroth and first power of z on both sides of equation (1) to be equal, solutions for the four expansion coefficients can be obtained. Utilizing the above expressions, equations (2) and (3) and assuming that Q = constant, the following equations result:

$$\int_0^\infty F_{1w}(e_{bw1} + e_{bw2} - 2e_{bw0}) dw = -3QL \tag{6}$$

$$\int_0^\infty \left[G_{2w}(e_{bw1} - e_{bw2}) + \left(G_{2w} + \frac{4}{3} G_{1w} \right) e_{bw0}^{(1)} L \right] dw = 0 \tag{7}$$

where $e_{bw0} = e_{bw}\ (z = 0)$ and $e_{bw0}^{(1)} = \partial e_{bw}/\partial z\ (z = 0)$. In the above equations, F_{kw} and G_{kw} are functions defined as

$$F_{kw} = \frac{6 L_w{}^k}{1 + L_w + 3 L_w{}^2/8} \tag{8}$$

$$G_{kw} = \frac{32 L_w{}^k}{L_w{}^3 + 4 L_w{}^2 + 8 L_w + 32/3} \tag{9}$$

with $L_w = a_w L$. For a given nongray medium with a known absorption coefficient a_w, equation (6) can be readily solved iteratively to yield the unknown centerpoint temperature $T(z = 0)$. Equation (7) can be similarly solved to generate $dT/dz\ (z = 0)$. These results can be used to determine the unknown expansion coefficients and subsequently the temperature profile and heat transfer. It is interesting to note that in the limit of a gray medium (a_w = const.), solutions to equations (6) and (7) correspond exactly to the traditional diffusion approximation.

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Contributed by The Heat Transfer Division and presented at the 19th AIChE/ASME National Heat Transfer Conference, Orlando, Fla., July 27–30, 1980. Revised manuscript received by The Heat Transfer Division April 21, 1980. Paper No. 80-HT-28.

3 Analytical Solution

Surprisingly, direct information on a_w for the different common absorbing gases (CO_2, H_2O, etc.) is scarce. Because of its highly irregular and complex behavior, most of the existing spectroscopic data on gaseous absorption are presented in terms of the correlation parameters for the wide-band total-absorptance model [5]. To generate an expression for a_w which is needed for the P-1 approximation, the present work proposes to utilize these data indirectly. Based on the Elsassar "narrow band" expression for a_w and evaluating at the center of the narrow band, the present work assumes that the absorption coefficient is given by

$$a_w = \left(\frac{\rho S}{d}\right) \frac{\sinh [2\pi\gamma/d]}{\cosh [2\pi\gamma/d] - 1} \tag{10}$$

where ρ is the gas density, S/d the mean-line-intensity-to-spacing ratio and γ/d is the line-width-to-spacing ratio for the considered narrow band. Over a wide band, it can be shown that S/d is a function of wave number and γ/d a function of temperature and pressure. They are tabulated for the various gases in reference [5].

Equations (6) and (7) can be further simplified by assuming that the blackbody emissive power e_{bw} varies only slowly over the range of w at which F_{kw} and G_{kw} are significantly different from zero. Treating e_{bw} as a constant in the integration, equations (6) and (7) become

$$\sum_{i=1}^{n} F_i^{(1)} (e_{b1i} + e_{b2i} - 2e_{b0i}) = -3QL \tag{6a}$$

$$\sum_{i=1}^{n} \left[G_i^{(2)} (e_{b1i} - e_{b2i}) + e_{b0i}{}^{(1)} L \left(G_i^{(2)} + \frac{4}{3} G_i^{(1)} \right) \right] = 0 \tag{7a}$$

where the subscript i stands for evaluation at the center (or the head) of the ith band w_i and $F_i^{(k)}$, $G_i^{(k)}$ are integrals of the function F_{kw} and G_{kw} over the ith band. It is interesting to note that utilizing equation (10), $F_i^{(k)}$ and $G_i^{(k)}$ can be evaluated in closed-form for all values of k.

Even without explicit numerical computations, a number of interesting analytical expressions can be readily generated. For the case with equal wall temperature ($T_1 = T_2$) and nonzero heat generation ($Q \neq 0$), assuming that the blackbody emissive power can be approximated by the following linearization

$$e_{b0i} = e_{b1i} + \left(\frac{de_{bi}}{dT}\right)_{T_1} (T_0 - T_1) \tag{14}$$

equation (10a) yields

$$T_0 = T_1 + \frac{\frac{3}{2} QL}{\sum_{i=1}^{n} F_i^{(1)} \left(\frac{de_{bi}}{dT}\right)_{T_1}} \tag{15}$$

n the optically thin limit, equation (21) is reduced to

$$T_0 = T_1 + \frac{QL}{4 \sum_{i=1}^{n} \left[\tau C_3 \left(\frac{de_b}{dT}\right)_{T_1} \right]_i} \tag{16}$$

where $\tau = \rho L (C_1/C_3) \sinh (2\pi\gamma/d)/(\cosh (2\pi\gamma/d) - 1)$, and C_1, C_3 he wide-band correlation constants tabulated in reference [5]. Equation (16) is the correct optically thin limiting expression for T_0 s discussed in reference [3]. In the large path-length limit ($\rho L \rightarrow$ arge), equation (15) is reduced to

$$T_0 = T_1 + \frac{QL}{6.96 \sum_{i=1}^{n} \left[C_3 \left(\frac{de_b}{dT}\right)_{T_1} \right]_i} \tag{17}$$

xcept for a simple numerical factor, equation (23) is again identical the corresponding expression developed in reference [3]. (The umerical value 6.96 is replaced by 2π in that reference.)

For the case with unequal wall temperatures ($T_1 \neq T_2$) and zero eat generation ($Q = 0$), the overall heat transfer is the more interesting physical quantity. Utilizing equations (6–9) and the definition of heat flux, it can be shown that in the large pathlength limit

$$q \rightarrow \sigma T_1^4 - \sigma T_2^4 - \sum_{i=1}^{n} (e_{b1i} - e_{b2i})(1n\tau)_i \tag{18}$$

Since $1n\tau$ is the large pathlength limit of the wide-band total gas absorptance, equation (18) suggests that in that limit the different bands absorb independently and their effects add linearly to yield the total attenuation. Since the contribution due to the different bands can be quite substantial, the above expression also illustrates that calculations in which only one absorption band is considered can lead to significant error in the heat flux prediction at the large pathlength limit.

4 Numerical Results and Discussion

To demonstrate quantitatively the effectiveness and the accuracy of the present method, solutions with CO_2 as the absorbing medium are now generated. Assuming that $Q = 0$ and the only relevant absorption band is the 4.3 μ band, solution to equations (10a) and (10b) yield the following expressions for the heat flux and the temperature distribution

$$q = \sigma T_1^4 - \sigma T_2^4 - (e_{b1c} - e_{b2c}) \left[\frac{1}{32} G_c^{(3)} + \frac{1}{8} G_c^{(2)} + \frac{1}{4} G_c^{(1)} \right] + [e_{b1c} - e_{b2c}) G_c^{(2)} \left[\frac{G_c^{(2)} + 4G_c^{(1)}}{24G_c^{(2)} + 32G_c^{(1)}} \right] \tag{19}$$

$$\frac{e_{bc}(z) - e_{b2c}}{e_{b1c} - e_{b2c}} = \frac{1}{2} - \frac{1}{4A_pL} \left[G_c^{(2)} \eta + \frac{1}{2} G_c^{(4)} \eta^3 \right] - \frac{G_c^{(2)} \left[G_c^{(4)} (\eta - 4\eta^3) + 4G_c^{(3)} \eta - \frac{16}{3} G_c^{(2)} \eta^3 \right]}{4A_pL \, [8G_c^{(2)} + 32/3 G_c^{(1)}]} \tag{20}$$

Table 1 Comparison of the dimensionless heat flux ($q/\sigma T_2^4$) for two typical cases obtained from the present P_1 approximation and those obtained from other techniques

	Case 1	Case 2
Wide-band Curtis-Godson method	195.029	189.481
Hottel's method	195.038	189.491
Method based on isothermal band absorption	194.887	189.727
P_1 approximation	194.1	189.2

Case 1: T_1 1500 K, T_2 = 400 K, P_{CO_2} = 0.2 atm, P_{N_2} = 10 atm, L = 1 cm.
Case 2: T_1 = 1500 K, T_2 = 400 K, P_{CO_2} = 1.0 atm, P_{N_2} = 0, L = 10 cm.

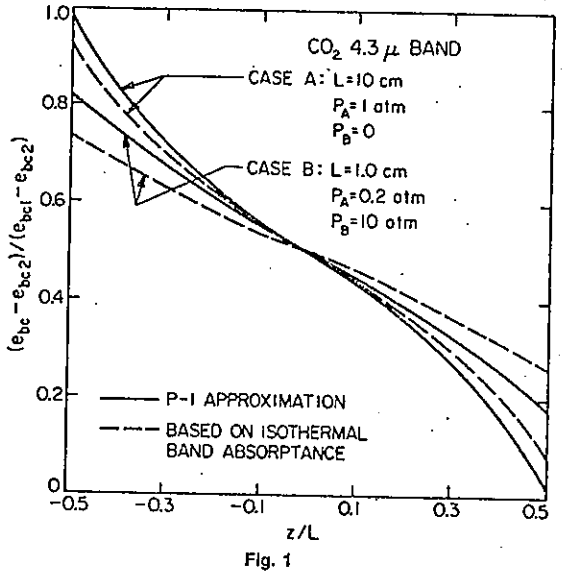

Fig. 1

In the above expression, $\eta = z/L$; the subscript c implies properties evaluated at the head of the 4.3 μ band and $A_p = 1/L \int_{\Delta_l} a_w L \, dw$ is the Planck mean absorption coefficient.

Assuming that $T_1 = 1500$ K and $T_2 = 400$ K, and evaluating $a_w q^+$ the average temperature, the heat flux calculated from equation (19) for two specific cases and those obtained from other techniques [2] are compared in Table 1. The agreement is excellent. The accuracy of the temperature profile result is illustrated in Fig. 1. It is important to note that the present mathematical development represents at least a ten-fold reduction in complexity comparing to all of the specific existing techniques.

Acknowledgment

This work is based upon work supported by the National Science Foundation Grant No. ENG78-05587.

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The Effects of Nonuniform Heat Transfer from an Annular Fin of Triangular Profile

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Nomenclature

$h = (h_b + h_t)/2$, average heat transfer coefficient
h_b = heat transfer coefficient at fin-base
h_t = heat transfer coefficient at fin-tip
k = thermal conductivity
r = radial distance from tube axis
r_b = radius of fin-base
r_t = radius of fin-tip
t = half thickness of fin-base
$x = (r_t - r)/L$, dimensionless distance from fin-tip
Bi $= ht/k$, Biot number
H = nonuniform heat transfer coefficient
L = fin length
$M = (L/t)^2 \text{Bi}/\cos\alpha$, dimensionless
$N = 2(1 + \xi)$, dimensionless
$R = r_t/L$, dimensionless
T_b = temperature of fin-base
T_∞ = temperature of fluid
$\alpha = \tan^{-1}(t/L)$
$\epsilon = h_b/h_t$, dimensionless
$\theta = (T - T_\infty)/(T_b - T_\infty)$, dimensionless temperature
$\xi = (1 - \epsilon)$, dimensionless

Introduction

Finned surfaces are presently designed on the basis that the heat transfer coefficient is uniform over the fin surface as this permits the use of well established analytic solutions [1–3]. However, experimental investigations indicate that the heat transfer coefficient is not invariant [4–6]. In particular, for the annular geometry the heat transfer coefficient varies both radially and angularly over the fin surface (Wong [4]).

Analytic investigation of the effects of nonuniform heat transfer involve the use of assumed variation of the heat transfer coefficient, e.g., Han and Lefkowitz [7] employ heat transfer coefficients which vary as given powers of the displacement from the fin-base.

In this note the analytic solution for the temperature distribution within an annular triangular fin (Fig. 1) with linearly varying heat transfer coefficient is derived, and the corresponding fin efficiency is compared with experimental results.

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Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division July 14, 1980.

Analysis

In the following analysis it is assumed that the thermal conductivity of the fin material, k, is constant and the heat transfer coefficient, H, varies linearly from h_b at the fin-base to h_t at the fin-tip, i.e.,

$$H(r) = \frac{2h}{(1+\epsilon)}\left(1 - (1-\epsilon)\frac{r_t - r}{r_t - r_b}\right) \tag{1}$$

where h is the average value of H and ϵ is the ratio of the heat transfer coefficient at the fin-base to that at the fin-tip.

For steady-state, one-dimensional conductive heat flow, an energy balance over an element of the fin, gives

$$\frac{d}{dr}\left(ry\frac{d}{dr}\theta\right) - \frac{H}{k\cos\alpha} r\theta = 0 \tag{2}$$

where the fin profile is described by

$$y(r) = \frac{t}{L}(r_t - r)$$

The inclusion of the exact representation for the incremental surface area, $(2\pi r\, dr/\cos\alpha)$, does not complicate the solution procedure, and has the advantage that the solution will be valid, even for small fin length to fin-base thickness ratios. This has particular significance as Lau and Tan [8] have recently shown, for a variety of fins, that the applicability of the one-dimensional approximation does not require the fin length to be large in comparison to the relevant transverse dimension.

The temperature distribution, and hence the heat transfer rate of the fin, are determined by solving equation (2) subject to the conditions

$$\text{at } r = r_b,\ \theta = 1 \tag{3}$$

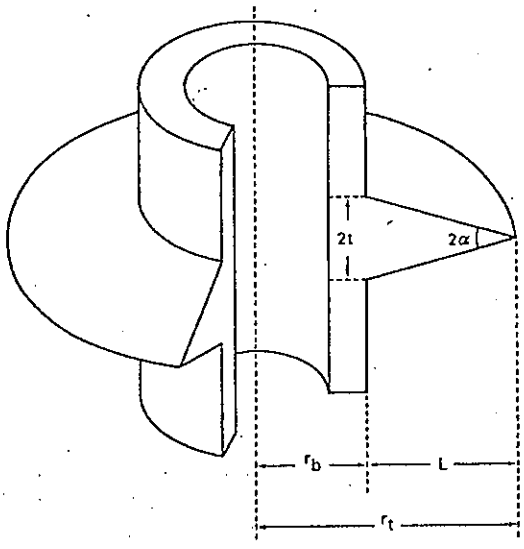

Fig. 1 The annular fin of triangular profile