Journal of Heat Transfer

Technical Notes

This section contains shorter technical papers. These shorter papers will be subjected to the same review process as that for full papers.

Non-Fourier Heat Conduction in a Semi-infinite Solid Subjected to Oscillatory Surface Thermal Disturbances

W. W. Yuen[1] and S. C. Lee[2]

1 Introduction

Many investigators have explored the effect of non-Fourier conduction in transient heat transfer processes in recent years (Brazel and Nolan, 1966; Maurer and Thompson, 1973; Kazimi and Erdman, 1975; Luikov, 1968; Wiggert, 1977; Glass et al., 1985). Based on a relaxation model for heat conduction in solids and liquids, the traditional heat diffusion equation is replaced with a hyperbolic equation that accounts for the finite thermal propagation speed. The use of the hyperbolic equation removes the nonphysical phenomenon of the diffusion equation analysis that predicts instantaneous temperature disturbances at all points in the medium for a step heat flux at the boundary. It further removes the peculiarity of an infinite temperature gradient at the boundary as time goes to zero.

The hyperbolic equation has been used in a number of analyses. Solutions generated by these studies show that the non-Fourier effect is important only at very early times in transient heat transfer processes, such as the high-intensity electromagnetic irradiation of a solid (Brazel and Nolan, 1966), the sudden contact of two hot molten liquids such as uranium dioxide and sodium (Kazimi and Erdman, 1975), and the high-rate heat transfer in rarefied media (Luikov, 1968). In general, the non-Fourier effect is shown to decay quickly, and the conventional Fourier equation is accurate a short time after the initial transient.

The objective of this work is to show that the non-Fourier conduction effect can be important even at a "long time" after the initial transient if the thermal disturbance is oscillatory with the period of oscillation of the same order of magnitude as the thermal relaxation time. In particular, the thermal response of a semi-infinite solid subjected to a sinusoidal boundary heat flux condition is generated. The present solution illustrates readily that, in many practical situations such as the repeated irradiation of a solid by a laser with very short pulse width, heat transfer analyses using the traditional Fourier heat diffusion equation can result in significant errors. The current results also suggest that the thermal relaxation time of a solid can be determined by measuring the thermal response of the solid irradiated by a high-frequency heat flux.

2 Analysis

In one-dimensional flow of heat, the energy equation is given by

$$\rho C \frac{\partial T}{\partial t} + \frac{\partial q}{\partial x} = 0 \tag{1}$$

where ρ is the solid density, C is the specific heat, T is the temperature, q is the heat flux, and x, t are the distance and time coordinates, respectively. The modified Fourier equation, as utilized in previous investigations (e.g., Brazel and Nolan, 1966), is

$$\tau \frac{\partial q}{\partial t} + q + k \frac{\partial T}{\partial x} = 0 \tag{2}$$

where k is the thermal conductivity and τ is defined as the thermal relaxation time which, for many solids, is on the order of 10^{-12} to 10^{-14} s (Weyman, 1967). Equations (1) and (2) can be combined to form the following dissipative wave equation:

$$\tau \frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \tag{3}$$

For a one-dimensional semi-infinite solid subjected to a sinusoidal surface heat flux boundary condition, the boundary and initial conditions are given by

$$\begin{aligned} q(0, t) &= q_0 e^{i\omega t} \\ T(\infty, t) &= 0 \\ q(\infty, t) &= 0 \end{aligned} \tag{4}$$

and

$$\begin{aligned} T(x, 0) &= 0 \\ q(x, 0) &= 0 \end{aligned} \tag{5}$$

where $i = (-1)^{1/2}$; q_0 and ω are amplitude and frequency of the surface heat flux oscillation, respectively. Note the initial temperature is normalized to be zero in the above equation. The sinusoidal surface heat flux is expressed as a complex number for mathematical convenience.

The solution to equation (3) can be readily generated by the Laplace transform. Taking the Laplace transform of equations (2) and (3) yields

$$(1 + \tau s)\bar{q}(x, s) = -k \frac{d\bar{T}}{dx}(x, s) \tag{6}$$

[1] Department of Mechanical and Environmental Engineering, University of California, Santa Barbara, CA 93106.

[2] Thermal Control Department, The Aerospace Corporation, El Segundo, CA 90245-4691.

Contributed by the Heat Transfer Division and presented at the National Heat Transfer Conference, Pittsburgh, Pennsylvania, August 9-12, 1987. Manuscript received by the Heat Transfer Division October 14, 1987. Keywords: Conduction, Transient and Unsteady Heat Transfer.

$$\frac{d^2\bar{T}(x, s)}{dx^2} = \frac{s}{\alpha}(1+\tau s)\bar{T}(x, s) \qquad (7)$$

where $\bar{q}$ and $\bar{T}$ are the Laplace transforms of T, respectively.

When equation (6) is evaluated at $x=0$, the boundary conditions become

$$\frac{d\bar{T}(0, s)}{dx} = -\left(\frac{1+\tau s}{s-i\omega}\right)\frac{q_0}{k}$$

$$\bar{T}(\infty, s) = 0 \qquad (8)$$

$$q(\infty, s) = 0$$

The solution to equation (7) subjected to the boundary conditions as represented by equation (8) is

$$\bar{T}(x, s) = \frac{q_0(\alpha\tau)^{1/2}}{k}\frac{(s+1/\tau)^{1/2}}{s^{1/2}(s-i\omega)}e^{-x/a[s(s+1/\tau)]^{1/2}} \qquad (9)$$

where

$$a = \left(\frac{\alpha}{\tau}\right)^{1/2}$$

is the thermal propagation speed. The inverse transform of equation (9) is

$$\frac{T(x, t)}{q_0(\alpha\tau)^{1/2}/k} = H(t-x/a)e^{-t/2\tau}I_0\left[\frac{(t^2-x^2/a^2)^{1/2}}{2\tau}\right]$$

$$+\left(\frac{1}{\tau}+i\omega\right)e^{i\omega t}\int_{x/a}^{t}e^{-(i\omega y+y/2\tau)}I_0\left[\frac{(y^2-x^2/a^2)^{1/2}}{2\tau}\right]dy \qquad (10)$$

where $H(t)$ is the Heaviside unit step function and $I_0(x)$ is the modified Bessel function of zero order.

3 Results and Discussion

(a) Temperature at x = at. The presence of the Heaviside function $H(t)$ in equation (10) illustrates that due to the finite speed of propagation of the thermal disturbance, the temperature in the solid remains at zero for $x>at$. At $x=at$, the temperature is independent of ω and has a step discontinuity given by

$$\frac{T(at, t)}{q_0(\alpha\tau)^{1/2}/k} = e^{-t/2t} \qquad (11)$$

At $t=0$, the above equation yields the surface temperature as

$$\frac{T(0, 0)}{q_0(\alpha\tau)^{1/2}/k} = 1$$

which is identical to results derived in Maurer and Thompson (1973) and Wiggert (1977). Equation (11) suggests that in the limit of short time (on the order of the thermal relaxation time), the temperature at $x=at$ is significantly higher than that predicted by the conventional Fourier analysis. At a sufficiently high heat flux level, this result has important implications in the failure of structural integrity due to thermal shock.

(b) "Steady-State" Results. In the limit of long time ($t\rightarrow\infty$), the integral in equation (10) can be evaluated in closed form. The temperature profile can be written analytically as

$$\frac{T(x,t)}{q_0(\alpha\tau)^{1/2}/k} = \frac{(1+\omega^2\tau^2)^{1/4}}{(\omega\tau)^{1/2}}e^{i(\omega t+\gamma)-(i+1)\kappa_+x-(i-1)\kappa_-x} \qquad (12)$$

where

$$\kappa_+ = \frac{1}{2}\left[\frac{\omega}{\alpha}((1+\omega^2\tau^2)^{1/2}+1)\right]^{1/2} \qquad (13a)$$

$$\kappa_- = \frac{1}{2}\left[\frac{\omega}{\alpha}((1+\omega^2\tau^2)^{1/2}-1)\right]^{1/2} \qquad (13b)$$

and

$$\gamma = \frac{1}{2}\tan^{-1}\omega\tau - \frac{\pi}{4} \qquad (14)$$

In the limit of $\tau\rightarrow 0$, equation (12) is reduced to

$$T(x, t) = \frac{q_0}{k\left(\frac{\omega}{\alpha}\right)^{1/2}}e^{i(\omega t-\pi/4)-(i+1)(\omega/2\alpha)^{1/2}x} \qquad (15)$$

which is identical to the result generated by conventional Fourier conduction analysis as given in Carslaw and Jaeger (1980).

Physically, equation (12) represents a decaying temperature wave of wavelength λ given by

$$\lambda = \frac{\lambda_0 2^{1/2}}{[(1+\omega^2\tau^2)^{1/2}+1]^{1/2}+[(1+\omega^2\tau^2)^{1/2}-1]^{1/2}} \qquad (16)$$

where

$$\lambda_0 = \frac{2\pi}{(\omega/2\alpha)^{1/2}} \qquad (17)$$

is the wavelength of the temperature wave calculated by the conventional Fourier conduction ($\tau=0$) analysis.

Equation (12) also shows that the amplitude of the temperature oscillation diminishes as $e^{-x/L}$ where

$$L = \frac{L_0 2^{1/2}}{[(1+\omega^2\tau^2)^{1/2}+1]^{1/2}-[(1+\omega^2\tau^2)^{1/2}-1]^{1/2}} \qquad (18)$$

with

$$L_0 = \left(\frac{2\alpha}{\omega}\right)^{1/2} \qquad (19)$$

Physically, L can be interpreted as a "steady-state" penetration depth for the surface thermal disturbance, while L_0 is the distance predicted by the conventional Fourier analysis.

In addition to the wavelength and penetration depth of the temperature wave, the non-Fourier prediction of the actual amplitude of the temperature oscillation is also significantly different from the Fourier prediction. Evaluating at $x=0$, it can be readily shown from equation (12) that the dimensionless surface temperature is given by

$$\frac{T(0, t)}{q_0(\alpha\tau)^{1/2}/k} = Ae^{i(\omega t+\gamma)} \qquad (20)$$

with

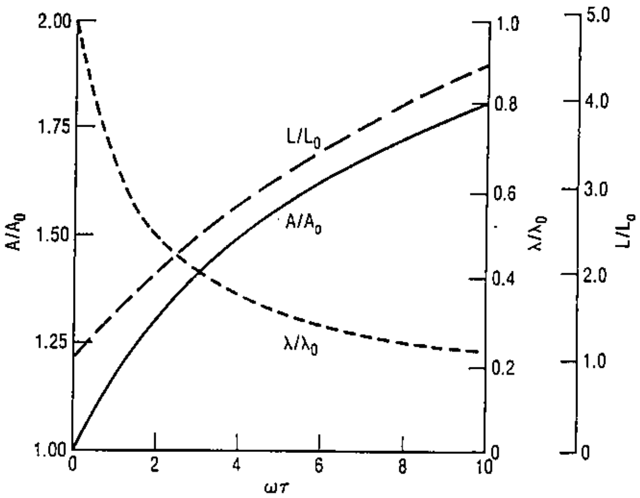
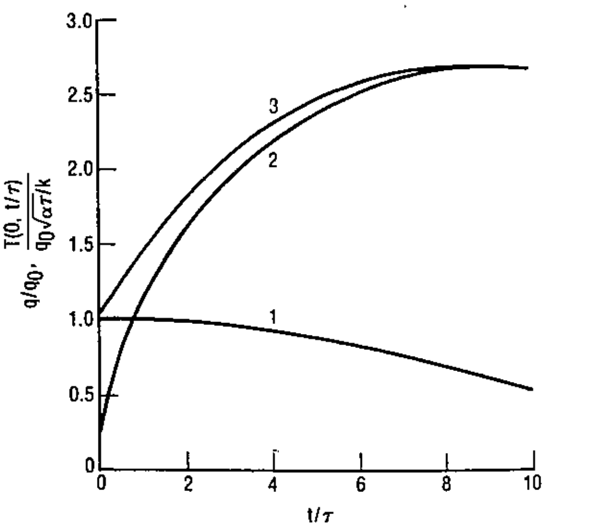
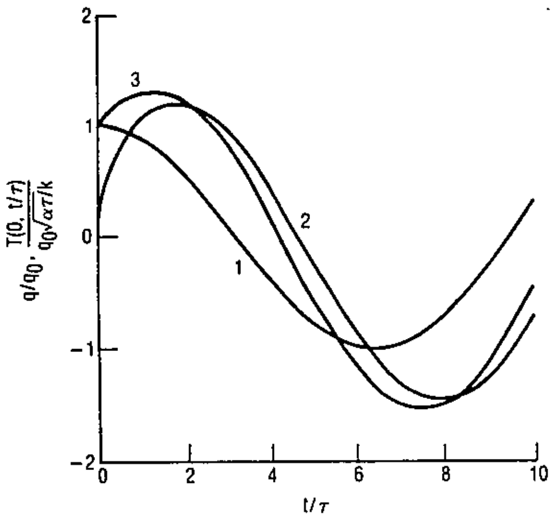


Fig. 1 Effect of non-Fourier conduction on the amplitude A/A_0, penetration depth L/L_0, and wavelength λ/λ_0 of the temperature response to a sinusoidal surface heat flux for steady state



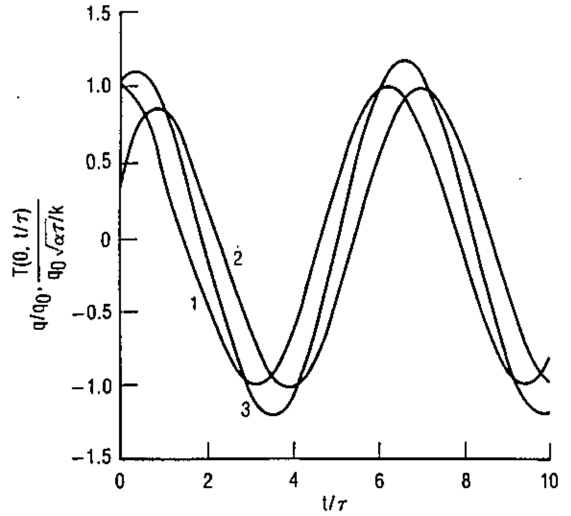
1 = DIMENSIONLESS SURFACE FLUX
2 = DIMENSIONLESS SURFACE TEMPERATURE (Fourier result)
3 = DIMENSIONLESS SURFACE TEMPERATURE (non-Fourier result)

Fig. 2 Transient surface temperature behavior for $\omega\tau = 0.1$



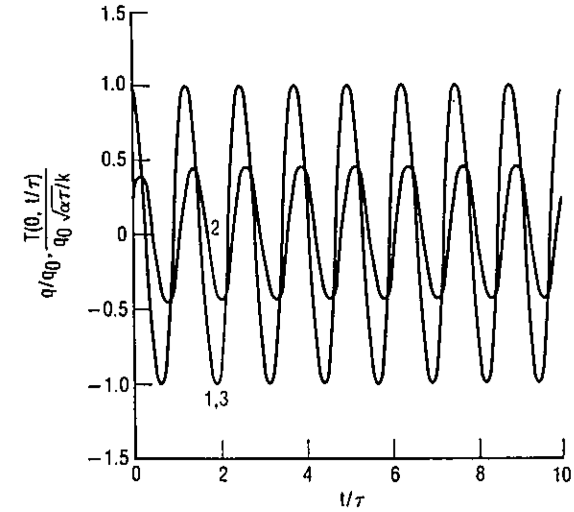
1 = DIMENSIONLESS HEAT FLUX
2 = DIMENSIONLESS SURFACE TEMPERATURE (Fourier result)
3 = DIMENSIONLESS SURFACE TEMPERATURE (non-Fourier result)

Fig. 3 Transient surface temperature behavior for $\omega\tau = 0.5$



1 = DIMENSIONLESS SURFACE FLUX
2 = DIMENSIONLESS SURFACE TEMPERATURE (Fourier result)
3 = DIMENSIONLESS SURFACE TEMPERATURE (non-Fourier result)

Fig. 4 Transient surface temperature behavior for $\omega\tau = 1.0$



1 = DIMENSIONLESS SURFACE FLUX
2 = DIMENSIONLESS SURFACE TEMPERATURE (Fourier result)
3 = DIMENSIONLESS SURFACE TEMPERATURE (non-Fourier result)

Fig. 5 Transient surface temperature behavior for $\omega\tau = 5.0$

$$A = \frac{(1+\omega^2\tau^2)^{1/4}}{(\omega\tau)^{1/2}} \tag{21}$$

The corresponding surface temperature predicted by the Fourier analysis, on the other hand, is given by Glass et al. (1985)

$$\frac{T(0,\,t)}{q_0(\alpha\tau)^{1/2}/k} = A_0 e^{i(\omega t - \pi/4)} \tag{22}$$

with

$$A_0 = (\omega\tau)^{-1/2} \tag{23}$$

It must be reiterated that equations (20) and (22) are for large t.

Plots of λ/λ_0, L/L_0, and A/A_0 against $\omega\tau$ are shown in Fig. 1. It is readily observed that the non-Fourier effect decreases the wavelength but increases the amplitude and penetration depth of the "steady-state" temperature wave. The effect is significant when the period of oscillation $1/\omega$ is less than or equal to the relaxation time constant τ. The non-Fourier effect thus has a "long time" permanent influence on the temperature response of a solid subjected to a high-frequency surface thermal disturbance.

It is noted that the product of the wavelength and penetration depth of the "steady-state" temperature wave is identical for both the Fourier and non-Fourier analysis. Specifically, the product varies inversely with the frequency of oscillation as

$$\lambda L = \lambda_0 L_0 = \frac{4\pi\alpha}{\omega} \tag{24}$$

In the limit of large frequency $\omega \to \infty$, the non-Fourier wavelength becomes

$$\lambda(\omega \to \infty) = \frac{2\pi}{\omega\tau}(\alpha\tau)^{1/2} \tag{25}$$

while the non-Fourier penetration depth is given by

$$L(\omega\to\infty)=2(\alpha\tau)^{1/2} \quad (26)$$

The corresponding Fourier penetration depth L_0, on the other hand, approaches zero as $\omega\to\infty$.

(c) General Transient Temperature Behavior. The surface temperature $T(0, t)$ for $0<t/\tau<10$ with different values of $\omega\tau$ (0.1, 0.5, 1.0, 5.0) was calculated based on equation (8) and presented in Figs. 2-5. The surface heat flux is assumed to be cosine varying. The corresponding Fourier prediction and dimensionless surface heat flux are plotted on the same figures for comparison. It should be pointed out that the surface heat flux q_0 is regarded as negative when q is negative. The negative temperature is due to the normalization by q_0.

For the case of small $\omega\tau$, γ approaches $\pi/4$ and the phase difference between the Fourier and non-Fourier diminishes. The non-Fourier effect appears only when t/τ is small, and the non-Fourier temperature profile becomes identical to the Fourier result when $t/\tau\geq 10$. This transient temperature behavior is consistent with the predictions of the previous works (e.g., Brazel and Nolan, 1966). For cases with moderate and large values of $\omega\tau$, however, the non-Fourier surface temperature differs significantly from the Fourier result for all values of t/τ. In the limit of large $\omega\tau$, γ appraoches zero and the non-Fourier surface temperature oscillates in phase with the surface heat flux. The Fourier surface temperature, on the other hand, has a phase shift of $\pi/4$ relative to the surface heat flux.

4 Conclusion

The non-Fourier thermal response of a solid subjected to an oscillatory surface heat flux has been considered in this paper. This corresponds to the practical situation of irradiation of a solid by a pulsed laser. Results show that, for moderate and large values of the oscillation frequency, the non-Fourier effect is quite significant. In contrast to many existing results, the non-Fourier effect is demonstrated to be important even for "long times" after the initial transient. Conventional Fourier conduction analysis is shown to underestimate the steady-state penetration depth of the thermal disturbance and overestimate the wavelength of the temperature propagation. In practical calculations that involve high-frequency surface thermal disturbances, such as the irradiation of a solid by a pulsed laser, an accurate thermal analysis should, therefore, include the non-Fourier effect. Results of the present work also suggest that measurement of the relative phase shift between the surface temperature and the applied heat flux would be an effective way to determine the thermal relaxation time of a solid.

References

Brazel, J. P., and Nolan, E. J., 1966, "Non-Fourier Effects in the Transmission of Heat," *Proceedings 6th Conference on Thermal Conductivity*, Dayton, OH, pp. 237-254.

Carslaw, H. S., and Jaeger, J. C., 1980, *Conduction of Solids*, 2nd ed., Oxford University Press, United Kingdom.

Glass, D. E., Ozisik, M. N., and Vick, B., 1985, "Hyperbolic Heat Conduction With Surface Radiation," *International Journal of Heat and Mass Transfer*, Vol. 28, No. 10, pp. 1823-1830.

Kazimi, M. S., and Erdman, C. A., 1975, "On the Interface Temperature of Two Suddenly Contacting Materials," ASME JOURNAL OF HEAT TRANSFER, Vol. 97, pp. 615-617.

Luikov, A. V., 1968, *Analytical Heat Diffusion Theory*, Academic Press, New York, pp. 245-248.

Maurer, M. J., and Thompson, H. A., 1973, "Non-Fourier Effects at High Heat Flux," ASME JOURNAL OF HEAT TRANSFER, Vol. 95, pp. 284-286.

Weymann, H. D., 1967, "Finite Speed of Propagation in Heat Conduction, Diffusion, and Viscous Shear Motion," *Am. Journal of Physics*, Vol. 35, pp. 488-496.

Wiggert, D. C., 1977, "Analysis of Early-Time Transient Heat Conduction by Method of Characteristics," ASME JOURNAL OF HEAT TRANSFER, Vol. 99, pp. 35-40.

1-2 *N* Shell-and-Tube Exchanger Effectiveness: a Simplified Kraus-Kern Equation

B. S. Baclic[1]

Nomenclature

- A = exchanger total heat transfer area on one side, m^2
- C = heat capacity rate = Wc_p, W/K
- c_p = specific heat at constant pressure, J/(kg K)
- N_{tu} = number of heat transfer units based on the tube-side heat capacity rate = UA/C_t
- P = temperature effectiveness of the tube-side stream = $(T_{t,o}-T_{t,i})$ / $(T_{s,i}-T_{t,i})$
- R = heat capacity rate ratio $=C_t/C_s$
- T = fluid temperature, °C
- U = overall heat transfer coefficient, W/(m^2 K)
- W = fluid mass flow rate, kg/s

Subscripts

- i = inlet to the exchanger
- o = outlet to the exchanger
- s = shell side
- t = tube side

Kraus and Kern (1965) derived the following P-N_{tu}-R relation for the heat exchangers with one shell pass and any even number n of tube passes:

$$P=2/\left\{1+R+\frac{2}{n}x\coth(xN_{tu}/n)+\frac{2}{n}f(z)\right\} \quad (1)$$

where

$$x=(1+n^2R^2/4)^{1/2} \quad (2)$$

$$z=\exp(2N_{tu}/n) \quad (3)$$

$$f(z)=(mz^m+(m-2)z^{m-1}+(m-4)z^{m-2}+\ldots -(m-4)z^2-(m-2)z-m)/(1+z+z^2+z^3+\ldots+z^m) \quad (4)$$

$$m=\frac{n}{2}-1 \quad (5)$$

Using a simple relation for the sum of a geometric progression Dodd (1982) derived a more convenient expression for $f(z)$

$$f(z)=\frac{m(z^{m+2}-1)-(m+2)z(z^m-1)}{(z-1)(z^{m+1}-1)} \quad (6)$$

This note is aimed at further simplification of the Kraus-Kern formula. It is sufficient to denote the even numbers

$$n=2N, \qquad N=1,2,\ldots \quad (7)$$

so that equation (3) becomes

$$z=\exp(N_{tu}/N) \quad (8)$$

and Dodd's expression, equation (6), yields

$$f(z)=N\frac{z^N+1}{z^N-1}-\frac{z+1}{z-1} \quad (9)$$

[1]University of Novi Sad, Tehnicki fakultet "Mihajlo Pupin," 23000 Zrenjanin, Yugoslavia.

Contributed by the Heat Transfer Division for publication in the JOURNAL OF HEAT TRANSFER. Manuscript received by the Heat Transfer Division November 17, 1987. Keywords: Heat Exhangers.