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NUMERICAL COMPUTATION OF AN IMPORTANT INTEGRAL FUNCTION IN TWO-DIMENSIONAL RADIATIVE TRANSFER

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Abstract—Analytical properties, series expansions and asymptotic expansions are generated for $S_n(x)$ which are important integral functions in the analysis of two-dimensional radiative transfer. These functions are shown to be Fourier transform of the generalized exponential integral function. Table of values of $S_1(x)$, $S_2(x)$, and $S_3(x)$ are presented.

1. INTRODUCTION

In the analysis of two-dimensional radiative transfer in an absorbing-emitting medium, the integral functions defined by

$$S_n(x) = \frac{2}{\pi}\int_1^\infty \frac{e^{-xt}\,dt}{t^n(t^2-1)^{1/2}},\ x \geq 0,\ n = 0, 1, 2, \ldots \tag{1}$$

are important. These functions, for example, were utilized by Smith in his analysis of isotropic scattering of radiation in a two-dimensional atmosphere.[1,2] In the numerical solution for two-dimensional radiative equilibrium utilizing Hottel's zonal method,[3] computation of $S_1(x)$, $S_2(x)$, and $S_3(x)$ is required. If a point allocation method is used for the same problem with the unknown temperature distribution approximated by polynomials,[4] evaluation of $S_n(x)$ for higher values of n is needed.

It is interesting to note that the integral function $S_n(x)$ is closely related to the generalized exponential integral $\epsilon_n(x, \beta)$. Introduced originally by Brieg and Crosbie[5–6] in their works on two-dimensional radiative equilibrium and generalized by Yuen and Wong,[4] $\epsilon_n(x, \beta)$ is defined by

$$\epsilon_n(\tau, \beta) \equiv \frac{\tau^{n-1}}{\pi}\int_{-\infty}^\infty \int_1^\infty \ldots \int_1^\infty e^{i\beta x_0} x_1^{\,n-1} \ldots x_{n-1}^1 K_0[x_1 \ldots x_n(x_0^{\,2} + \tau^2)^{1/2}\,dx_n \ldots dx_0, \tag{2}$$

where $K_0(x)$ is the modified Bessel function. It can be readily shown that $S_n(x)$ and $\epsilon_n(x, \beta)$ form an almost Fourier-transform pair, i.e.,

$$\epsilon_n(\tau, \beta) = \frac{\tau^{n-1}}{2}\int_{-\infty}^\infty \frac{e^{i\beta x} S_n[(x^2+\tau^2)^{1/2}]}{(x^2+\tau^2)^{n/2}}\,dx \tag{3}$$

and

$$S_n(2^{1/2}\tau) = \frac{2^{n/2}\tau}{\pi}\int_{-\infty}^\infty e^{-i\beta\tau}\epsilon_n(\tau, \beta)\,d\beta. \tag{4}$$

Evaluation of $S_n(x)$ is thus equivalent to the evaluation of $\epsilon_n(x, \beta)$.

The objective of the present work is to present some basic analytical properties of the integral function $S_n(x)$. Series expansion of $S_n(x)$ that is convergent for small values of x will be developed. Asymptotic expansions at large x and suitable for numerical computation will be presented. An interesting recursive relation for $\epsilon_n(x, \beta)$ will also be introduced. Since a direct numerical integration of Eq. (1) is difficult due to the existence of singularity, numerical values of $S_1(x)$, $S_2(x)$ and $S_3(x)$ generated by using series expansion and asymptotic expansion will be presented. An alternate expression for $S_n(x)$ has been utilized by various investigators[3,7,8] in studies on two-dimensional radiative equilibrium. The validity of this alternate expression and its shortcoming will be discussed.

2. BASIC PROPERTIES AND SERIES EXPANSIONS OF $S_n(x)$.

From the basic definition presented by Eq. (1), it can be shown that

$$S_n(0) = \frac{1}{\pi^{1/2}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}, \tag{5}$$

$$S_0(x) = \frac{2}{\pi} K_0(x), \tag{6}$$

$$\frac{\mathrm{d}S_n(x)}{\mathrm{d}x} = -S_{n-1}(x), \quad n \geq 1, \tag{7}$$

$$S_n(x) < S_m(x) \quad n > m, \tag{8}$$

and

$$S_n(x) < S_n(y) \quad x > y. \tag{9}$$

In Eq. (5), $\Gamma(x)$ is the gamma function. In the analysis of two-dimensional radiative equilibrium, two alternate expressions for $S_n(x)$ often appear. These are

$$S_n(a\tau) = \frac{\tau^n}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{-a(x^2+\tau^2)^{1/2}}}{(x^2+\tau^2)^{\frac{n+1}{2}}} \mathrm{d}x, \quad a, \ \tau \geq 0, \ n = 0, 1, 2, \ldots \tag{10}$$

and

$$S_n(x) = \frac{2}{\pi} \int_0^{\pi/2} \mathrm{e}^{-x \sec\theta} \cos^{n-1}\theta \, \mathrm{d}\theta. \tag{11}$$

Equation (10) appears in the Hottel zonal method of solution and Eq. (11) is used in the work of Glatt and Olfe[7] and Modest.[8] A set of relations analogous to Eqs. (5)–(9) can be readily generated for $\epsilon_n(x, \beta)$. The three most useful relations are

$$\epsilon_n(x, o) = E_n(x), \tag{12}$$

$$\epsilon_n(\tau, \beta) = \tau \int_1^{\infty} \epsilon_{n-1}(\tau x, \beta/x) \, \mathrm{d}x, \quad n \geqslant 1 \tag{13}$$

and the recursive relation

$$(n-1)\epsilon_n(\tau, \beta) = -\tau\epsilon_{n-1}(\tau, \beta) - \tau^{n-2} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{\epsilon_{n-2}(t, \beta)}{\tau^{n-3}}\right], \quad n \geq 2. \tag{14}$$

In Eq. (12), $E_n(x)$ is the exponential function. The proof of Eq. (14) is presented in Ref. 4.

Utilizing Eqs. (5)–(7), it can be shown that $S_n(x)$ may be represented by the Maclaurin expansion

$$S_n(x) = \frac{1}{\pi^{1/2}} \sum_{k=0}^{n-1} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)} \frac{(-x)^k}{k!} + \frac{2}{\pi}(-x)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(x/2)^{2k}}{(2k+n)!} \cdot \left(\psi_{k+1} - \ln\frac{x}{2} + \sum_{m=0}^{n-1} \frac{1}{2k+m+1}\right), \tag{15}$$

where

$$\psi_n = -\gamma \text{ as } n = 1 \text{ and } -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \text{ as } n > 1 \tag{16}$$

with γ being the Euler constant. The detailed development of Eq. (15) is given in the Appendix. Utilizing the root test,[9] it can be shown that the series converges for all values of x.

For large x, the Watson lemma[10] can be used to yield the asymptotic series

$$S_n(x) \sim \frac{e^{-x}}{\pi}\left(\frac{2}{x}\right)^{1/2} \sum_{k=0}^{\infty}(-1)^k \sum_{m=0}^{k} \frac{\binom{2m}{m}}{8^m}\binom{k-m+n-1}{n-1} \frac{\Gamma\left(\frac{2k+1}{2}\right)}{x^k}. \tag{17}$$

The detailed development of Eq. (17) is also given in the Appendix.

4. NUMERICAL COMPUTATION

Numerical values for $S_1(x)$, $S_2(x)$, and $S_3(x)$ are listed in Table 1. The results are accurate to six significant figures. For $0 \le x \le 2.0$, Eq. (15) is utilized. For $x \ge 20$, Eq. (17) is observed to be accurate and has been used for numerical computation. For $2.0 < x < 20$, results are generated by direct numerical integration of Eq. (11).

Table 1. Numerical values of $S_1(x)$, $S_2(x)$, and $S_3(x)$.

x	$S_1(x)$	$S_2(x)$	$S_3(x)$
0.0	1.0	0.636620	0.5
1.E-5	0.999920	0.636610	0.499994
1.E-3	0.994892	0.635622	0.499364
1.E-2	0.963578	0.626818	0.493683
0.1	0.782171	0.549098	0.440887
0.2	0.651695	0.477757	0.389652
0.3	0.553116	0.417717	0.344961
0.4	0.474411	0.366477	0.305817
0.5	0.409788	0.322368	0.271428
0.6	0.355801	0.284166	0.241146
0.7	0.310160	0.250930	0.314430
0.8	0.271239	0.221910	0.190820
0.9	0.237828	0.196499	0.169927
1.0	0.208994	0.174192	0.151417
2.0	6.18289E-2	5.44250E-2	4.89965E-2
5.0	2.17020E-3	2.02342E-3	1.90110E-3
7.0	2.54681E-4	2.41223E-4	2.29578E-4
10.0	1.08322E-5	1.03998E-5	1.00129E-5
15.0	6.06368E-8	5.89126E-8	5.73220E-8
20.0	3.57068E-10	3.49175E-10	3.41767E-10
25.0	2.16404E-12	2.12490E-12	2.08774E-12
30.0	1.33616E-14	1.31571E-14	1.29614E-14
35.0	8.35821E-17	8.24727E-17	8.14053E-17
40.0	5.27901E-19	5.21717E-19	5.15740E-19
45.0	3.35904E-21	3.32382E-21	3.28967E-21
50.0	2.14999E-23	2.12959E-23	2.10975E-23

It is interesting to note that direct numerical integration by quadrature of Eqs. (10) and (11), a procedure used in Refs. 3, 7, and 8, cannot yield accurate results in the optically thin limits (small a and x). This fact accounts for many of the difficulties encountered in existing numerical solutions for two-dimensional radiative equilibrium. Equation (15) is accurate in the limit of small x.

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APPENDIX

To derive Eq. (15), we consider the related function

$$F_n(x) = S_n(x) + \frac{2}{\pi}(-x)^n \ln x \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(x/2)^{2k}}{(2k+n)!}. \tag{18}$$

It can be shown that the infinite series in Eq. (18) converges for all values of $x \geq 0$. Using Eqs. (5)–(7), we find

$$F_n^{(k)}(0) = \frac{(-1)^k}{\pi^{1/2}} \frac{\Gamma\left(\frac{n-k}{2}\right)}{\Gamma\left(\frac{n-k+1}{2}\right)}, \quad k = 0, 1, \ldots, n-1 \tag{19}$$

To calculate $F_n^{(k)}(0)$ for $k \geq n$, Eqs. (5)–(7) and (18) are first combined to

$$F_n^{(n)}(x) = (-1)^n \frac{2}{\pi} K_0(x) + \frac{2}{\pi}(-1)^n \sum_{k=0}^{\infty} \binom{2k}{k} \frac{(x/2)^{2k}}{(2k)!} \left(\ln x + \sum_{m=0}^{n-1} \frac{1}{2k+m+1}\right). \tag{20}$$

Expanding $K_0(x)$ as a MacLaurin series, Eq. (20) becomes

$$F_n^{(n)}(x) = (-1)^n \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2} \left(\sum_{m=0}^{n-1} \frac{1}{2k+m+1} + \ln 2 - \gamma\right). \tag{21}$$

Differentiating Eq. (21) successively and evaluating at $x = 0$ yields

$$F_n^{(2k+n+1)}(0) = 0, \quad k = 0, 1, \ldots \tag{22}$$

and

$$F_n^{(2k+n)}(0) = \frac{(-1)^n}{2^{2k-1}\pi} \binom{2k}{k} \left(\psi_{k+1} + \ln 2 + \sum_{m=0}^{n+1} \frac{1}{2k+m+1}\right) k = 0, 1, \ldots \tag{23}$$

The Maclaurin series of $F_n(x)$ is

$$F_n(x) = \sum_{k=0}^{\infty} F_n^{(k)}(0) \frac{x^{(k)}}{k!}. \tag{25}$$

Utilizing Eqs. (19), (22), and (23), Eq. (15) results.

To develop Eq. (17) we set $t = 1 + y^2/2$, Eq. (1) is then transformed to

$$S_n(x) = \frac{2\,\mathrm{e}^{-x}}{\pi} \int_0^{\infty} \mathrm{e}^{-(x/2)t^2} \left(1 + \frac{t^2}{2}\right)^{-n} \left(1 + \frac{t^2}{4}\right)^{-1/2} \mathrm{d}t. \tag{26}$$

Expanding the nonexponential part of the integral as a double series, Eq. (26) becomes

$$S_n(x) \sim \frac{2\,\mathrm{e}^{-x}}{\pi} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+k}}{2^{4m+k}} \binom{2m}{m} \binom{n+k-1}{k} \int_0^{\infty} \mathrm{e}^{-(x/2)t^2 t^{2(m+k)}}\,\mathrm{d}t. \tag{27}$$

Integrating Eq. (27) and rearranging the indexes, Eq. (17) is obtained.