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Heat Transfer by Conduction and Radiation in a One-Dimensional Absorbing, Emitting and Anisotropically-Scattering Medium

W. W. Yuen
Assistant Professor.

L. W. Wong
Graduate Student.

Department of Mechanical and Environmental Engineering,
University of California,
Santa Barbara, Calif. 93106

Heat transfer by simultaneous conduction and radiation in an absorbing, emitting and anisotropically-scattering material is investigated theoretically. Consideration is given to a one-dimensional system bounded by two parallel gray, diffuse and isothermal walls. Assuming a physical model of linear-anisotropic scattering, the resulting integral-differential equation is solved by a successive approximation technique similar to the method of undetermined parameters. The solution method is demonstrated to be relatively simple and yields solution converging qucikly to the exact results. Results show that for the present one-dimensional system, the common approach of treating the total heat transfer as a simple addition of separate independent contributions from conduction and radiation is quite inaccurate for certain cases. This approach is thus ineffective in illustrating the general effect of scattering. Both the scattering albedo and the forward-backward scattering parameters are shown to have some interesting effects on the total heat transfer and the medium's temperature. The magnitude of these effects depends on the surface emissivity of the two boundaries.

Introduction

Heat transfer by simultaneous conduction and radiation between two reflecting surfaces with an intervening medium capable of absorbing, emitting and scattering thermal radiation is a problem of considerable practical importance. It serves as the basis, for example, in the analysis of the thermal performance of porous insulating materials such as fibers, powders, foams and many others. A great deal of work have been reported in this area [1–8], but most of them have only limited success largely because of the complexity of the problem.

Mathematically, the problem of simultaneous conduction and radiation is quite formidable as it involves the complex interaction between the radiative properties of the boundary surface and the thermal and optical properties of the material. The analysis requires a difficult solution of a nonlinear integral-differential equation. Many approximation methods have been proposed [1–4]. But nearly all of these methods were successful only in generating limiting expressions for the total heat flux. Few can predict the temperature profile accurately and none of them consider the effects of anisotropic scattering. Numerically, the only successful solution appears to be that of Viskanta and Grosh [6, 7]. Using an iterative method, they analyzed numerically the problem of combined conduction radiation in a one-dimensional absorbing, emitting, but nonscattering medium bounded by two parallel gray isothermal surfaces. The effect of the various system parameters on heat transfer were established and the calculation was later generalized to include the effect of isotropic scattering [8]. But the effect of anisotropic scattering is again neglected. While it is well known that the scattering of thermal radiation by real particles is by no means isotropic [9] and that anisotropic scattering can play a significant role on the overall heat transfer, all of the existing work on anisotropic-scattering media [10–12] avoid much of the mathematical complexities by considering only the effect of radiative transfer. None of them consider the simultaneous effect of conduction.

The objective of the present work is to obtain accurate solutions to the problem of simultaneous conduction and radiation in an absorbing emitting medium with anisotropic scattering. Assuming a model of linear-anisotropic scattering, successive approximate solutions converging to the exact result are generated. The solution method is similar to a technique which was utilized successfully in some recent analyses of radiative transfer [10, 11]. Unlike those analyses, the present method does not assume a differential formulation for the radiative intensity. Instead, the governing integral-differential equation is solved exactly. The difficult question concerning the convergence of the differential formulation is thus avoided. As in the previous approaches, the governing equation and its associated boundary conditions are reduced to a set of nonlinear algebraic equations at each step of the successive approximations. Solutions are obtained quickly and efficiently by simple iterations. In the limit of pure radiation, the present technique is identical to the method of undetermined parameters [5].

Results indicate clearly the effect of various system parameters on the heat transfer and the temperature profile of the medium. The common practice of treating the total heat transfer as a sum of separate independent contributions from conduction and radiation is demonstrated to be inappropriate except for some special limiting situations. In general, scattering can have an important effect on the total heat transfer and the medium's temperature profile. In some instances, differences between the anisotropic-scattering result and the isotorpic-scattering result are quite significant.

Mathematical Formulation. The physical system chosen and its associated coordinate system for the present analysis are identical to those considered in references [11] and [12]. Utilizing the same set of physical assumptions, the equation of transfer may be written as

$$\frac{di}{d\tau} + i = (1-\omega_0)i_b + \frac{\omega_0}{2}\int_{-1}^{1} i d\mu + \frac{\omega_0 x}{2}\int_{-1}^{1} i\mu d\mu \qquad (1)$$

where i denotes the radiation intensity, i_b the blackbody intensity, ω_0 the single scattering albedo, x the forward-backward scattering parameter, $\mu \equiv \cos\theta$ and τ is the optical thickness defined by

$$d\tau = \beta\, dz \qquad (2)$$

with z being the axis of symmetry and β the extinction coefficient. For combined conduction radiation, the energy equation is

$$\frac{d}{d\tau}\left[2\tau \int_{-1}^{1} i\mu d\mu - K\beta\frac{dT}{d\tau}\right] = \frac{H}{\beta} \qquad (3)$$

where k is the thermal conductivity of the medium, T the temperature

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and H is the internal heat generation rate.

Introducing the following dimensionless variables:

$$\theta = \frac{T}{T_1}, \theta_H^4 = \frac{H}{\beta\sigma T_1^4}, I = \frac{\pi i}{\sigma T_1^4}, G = 2\int_{-1}^{1} I d\mu, Q = 2\int_{-1}^{1} I\mu d\mu \quad (4)$$

where T_1 is the temperature of the lower wall and σ is the Stefan-Boltzmann constant, equations (1) and (3) can be written as

$$\mu\frac{dI}{d\tau} + I = (1-\omega_0)\theta^4 + \frac{\omega_0}{4}G + \frac{\omega_0 x\mu}{4}Q \quad (1a)$$

$$\frac{d}{d\tau}\left[Q - 4N_1\frac{d\theta}{d\tau}\right]\theta_H^4 \quad (3a)$$

where $N_1 = k\beta/4\sigma T_1^3$ is the familiar conduction-radiation parameter.

Equation (1a) can be integrated over μ to give

$$\frac{1}{2}\frac{dQ}{d\tau} + \frac{1}{2}G = 2(1-\omega_0)\theta^4 + \frac{\omega_0}{2}G \quad (5)$$

Together with equation (3a), the dimensionless average intensity G can be expressed as

$$G = 4\theta^4 - \left[\theta_H^4 + 4N_1\frac{d^2\theta}{d\tau^2}\right]/(1-\omega_0) \quad (6)$$

Assuming a constant heat generation rate, equation (3a) can be integrated over τ to yield

$$Q = 4N_1\frac{d\theta}{d\tau} + \theta_H^4\tau - 4N_1\left(\frac{d\theta}{d\tau}\right)_0 + Q_0 \quad (7)$$

Utilizing equation (6) and (7), equation (1a) becomes,

$$\mu\frac{dI}{d\tau} + I = S(\tau,\mu) \quad (8)$$

where

$$S(\tau,\mu) = \theta^4 - \left(\frac{\omega_0}{1-\omega_0}\right)\left(N_1\frac{d^2\theta}{d\tau^2} + \frac{\theta_H^4}{4}\right) + \omega_0 x\mu\left[N_1\frac{d\theta}{d\tau} + \frac{\theta_H^4}{4} - N_1\left(\frac{d\theta}{d\tau}\right)_0 + \frac{Q_0}{4}\right] \quad (9)$$

Formally, the solution to equation (8) is

$$I_+ = I_+\left(-\frac{L}{2}\right)e^{(-(\tau+L/2)/\mu)} + \int_{-\frac{L}{2}}^{\tau} S(\tau^*,\mu)e^{-((\tau-\tau^*)/\mu)}\frac{d\tau^*}{\mu}, \; 0 \le \mu \le 1 \quad (10a)$$

$$I_- = I_-\left(\frac{L}{2}\right)e^{-(\tau-L/2/\mu)} - \int_{\tau}^{\frac{L}{2}} S(\tau^*,\mu)e^{((\tau^*-\tau)/\mu)}\frac{d\tau^*}{\mu}, \; -1 \le \mu \le 0 \quad (10b)$$

Equations (6, 9, 10a, 10b) and the definition of G can be combined to give the following integral-differential equation for the dimensionless temperature θ:

$$\theta^4 - \frac{N_1}{1-\omega_0}\frac{d^2\theta}{d\tau^2} - \frac{1}{2}\int_0^1\left[\int_{-\frac{L}{2}}^{\tau} S(\tau^*,\mu)e^{-((\tau-\tau^*)/\mu)} \times d\tau^* + \int_{\tau}^{\frac{L}{2}} S(\tau^*,-\mu)e^{-(\tau^*-\tau/\mu)}d\tau^*\right]\frac{d\mu}{\mu}$$
$$= \frac{1}{2}\left[I_+\left(\frac{-L}{2}\right)E_2\left(\frac{L}{2}+\tau\right) + I_-\left(\frac{L}{2}\right)E_2\left(\frac{L}{2}-\tau\right)\right] + \frac{\theta_H^4}{4} + \frac{\omega_0\theta_H^4}{8(1-\omega_0)}\left[E_2\left(\frac{L}{2}+\tau\right) + E_2\left(\frac{L}{2}-\tau\right)\right]$$
$$+ \frac{\omega_0 x}{2}\left[\frac{Q_0}{4} - N_1\left(\frac{d\theta}{d\tau}\right)_0\right]\left[E_3\left(\frac{L}{2}+\tau\right) - E_3\left(\frac{L}{2}-\tau\right)\right] + \frac{\omega_0 x\theta_H^4}{8}\left\{-\frac{2}{3} + \frac{L}{2}\left[E_3\left(\frac{L}{2}+\tau\right) + E_3\left(\frac{L}{2}-\tau\right)\right] + E_4\left(\frac{L}{2}+\tau\right) + E_4\left(\frac{L}{2}-\tau\right)\right\} \quad (11)$$

where $E_n(x)$ is the exponential-integral function defined by

$$E_n(x) = \int_0^1 \mu^{n-2}e^{-x/\mu}d\mu \quad (12)$$

Utilizing equations (10a) and (10b) again, the constant θ_0, $I_+(-L/2)$ and $I_-(L/2)$ in equation (11) can be expressed in terms of θ as

$$\theta_0\left\{1 - \omega_0 x\left[\frac{1}{3} - E_4\left(\frac{L}{2}\right)\right]\right\} = 2E_3\left(\frac{L}{2}\right)\left(I - \theta_2^4\right) - 4N_1\omega_0 x\left[\frac{1}{3} - E_4\left(\frac{L}{2}\right)\right]\left(\frac{d\theta}{d\tau}\right)_0$$
$$+ 2\int_0^1\left[\int_{-\frac{L}{2}}^{0} S(\tau^*,\mu)e^{\tau^*/\mu}d\tau^* - \int_0^{\frac{L}{2}} S(\tau^*,-\mu)e^{-\tau^*/\mu}d\tau^*\right]d\mu \quad (13)$$

$$I_+\left(-\frac{L}{2}\right)[1 - 4(1-\epsilon_1)(1-\epsilon_2)E_3^2(L)] = \epsilon_1 + 2(1-\epsilon_1)\left\{\epsilon_2 E_3(L)\theta_2^4 + \int_0^1\int_{-\frac{L}{2}}^{\frac{L}{2}}\left[S(\tau^*,-\mu)e^{-L/2-\tau^*/\mu} + 2(1-\epsilon_2)S(\tau^*,\mu)e^{-(L/2+\tau^*)/\mu}\right]d\tau^* d\mu\right\} \quad (14)$$

Nomenclature

A_n = expansion coefficient defined by equation (19)
B_n = expansion coefficient defined by equation (20)
d = thickness of the one-dimensional slab
G = dimensionless average intensity defined by equation (4)
E_N = exponential function defined by equation (12)
H = internal heat generation rate
i = radiative intensity
i_b = blackbody intensity
I = dimensionless radiative intensity defined by equation (4)
J = Jacobian matrix
k = thermal conductivity
$L = \beta d$
M_1 = new conduction-radiation parameters N_1/L^2
N_1 = conduction-radiation parameter, $k\beta/4 T_1^3$
Q = dimensionless radiative heat flux defined by equation (4)
Q_0 = Q evaluated at $\tau = 0$
Q_t = total heat flux defined by equation (22)
S = source function defined by equation (9)
T = temperature
T_1 = temperature of the lower wall
x = forward-backward scattering parameter
$X_{i,j}$ = matrix element defined in equation (21)
$Y_{i,j}$ = matrix element defined in equation (21)
Z_i = vector defined in equation (21)
z = spacial coordinate
β = extinction coefficient
ϵ = surface emissivity
θ = dimensionless temperature, T/T_1
θ_m = θ evaluated at τ_m
$\theta_H = H/\beta\sigma T_1^4$
$\theta_0 = \theta$ at $\tau = 0$
θ_2 = dimensionless upper wall temperature
$\mu = \cos\theta$
τ = optical thickness defined by equation (2)
ω_0 = scattering albedo

$$I_-\left(\frac{L}{2}\right)[1-4(1-\epsilon_1(1-\epsilon_2)E_3^2(1)]$$

$$= \epsilon_2\theta_2^4 + 2(1-\epsilon_2)\left\{\epsilon_1 E_3(L)\theta_2^4 + \int_0^1 \int_{-\frac{L}{2}}^{\frac{L}{2}} \left[S(\tau^*, \mu)e^{-(L/2+\tau^*)/\mu} + 2(1-\epsilon_1)S(\tau^*, -\mu)e^{-(L/2-\tau^*)/\mu}\right] d\tau^* d\mu\right\} \quad (15)$$

The boundary conditions for θ are

$$\theta\left(-\frac{L}{2}\right) = 1, \theta\left(\frac{L}{2}\right) = \theta_2 \quad (16)$$

Equations (11) and (13–15) constitute a complete mathematical description for the present one-dimensional problem.

Method of Solution. Analytical solution to equations (11) and (16) are clearly impossible to obtain. Numerically, Viskanta [6–8] had demonstrated that for cases with no-scattering or isotropic scattering, solutions can be generated by a direct iterative method. While this same technique will probably be effective also for the case with anisotropic scattering, the present work chooses to use a different solution procedure. Unlike the iterative method, each solution in the present successive approximation series is developed independently. At each step of the approximation, equation (11) is reduced to a set of finite non-linear algebraic equations which can be readily solved. In contrast to the direct iterative method, the present method has the advantages that it is mathematically simple and solutions generated converge quickly independent of values of the various system parameters.

Mathematically, the present solution method is based on a simple observation that if θ and θ^4 are expressed as power series, the various integrals appeared in equations (11) and (14–15) can be carried out analytically in terms of the exponential integral function. Evaluating equation (11) at different values of τ, a system of algebraic equations can be generated to determine the unknown expansion parameters. From a computational point of view, these equations will have the same degree of complexity as a finite-difference formulation of a nonlinear ordinary differential equation. They can thus be readily solved.

In the Jth approximation, the unknown parameters are assumed to be the temperature at $2J + 1$ distinct locations in the medium as follows:

$$\theta_m = \theta(\tau_m) \quad (17)$$

where

$$\tau_m = \frac{mL}{2(J+1)} m = 0, \pm 1, \ldots, \pm J \quad (18)$$

In terms of the τ_m's, the temperature profile is expanded as a $(2J + 3)$ term power series. This gives

$$\theta = \sum_{n=0}^{2J+2} A_n \tau^n \quad (19)$$

Evaluating equation (19) at τ_m's and at the two boundaries, the coefficients A_n's can be expressed as linear functions of τ_m's. In a similar manner, θ^4 is approximated by

$$\theta^4 = \sum_{n=0}^{2J} B_n \tau^n \quad (20)$$

But unlike equation (19), equation (20) is not required to satisfy the noslip temperature boundary conditions. Results show that this relaxing of the temperature boundary condition for θ^4 greatly improves the accuracy of the solution for the lower order approximation. This is not surprising because in the limit of a large radiative effect, a θ^4 distribution which satisfies the no-slip temperature boundary condition will always yield an inaccurate radiative heat flux prediction in the optically thin limit. In the higher order approximation ($J \to \infty$), equations (19) and (20) clearly reduce to two consistent relations. By assuming θ and θ^4 to be separate power series, the present work also reduces greatly the complexity of the computation since only θ_m and $\theta_m{}^4$ will appear in the final matrix equation. If only equation (18) is utilized, all possible products of τ_m's up to the fourth power will appear. In all considered cases, the accuracy of the solution generated by the present method is better than or at least equal to those generated by assuming a simple power series for θ and θ^4. The computational time is always much less.

Substituting equations (19) and (20) into equation (11) and evaluated at τ_m's, a nonlinear matrix equation of the following general form results.

$$\begin{bmatrix} X_{1,1} & \cdots & X_{1,2J+1} \\ \vdots & & \vdots \\ X_{2J+1,1} & \cdots & X_{2J+1,2J+1} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_{-1} \\ \vdots \\ \theta_J \\ \theta_{-J} \end{bmatrix} + \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,2J+1} \\ \vdots & & \vdots \\ Y_{2J+1,1} & \cdots & Y_{2J+1,2J+1} \end{bmatrix} \begin{bmatrix} \theta_0^4 \\ \theta_1^4 \\ \theta_{-1}^4 \\ \vdots \\ \theta_J^4 \\ \theta_{-J}^4 \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_{2J} \\ Z_{2J+1} \end{bmatrix} \quad (21)$$

The solution can be generated by any common iterative technique.

Results and Discussion. Because of the large number of physical parameters involved even for a simple one-dimensional problem, the present consideration will be limited to those cases with $\epsilon_1 = \epsilon_2 = \epsilon$, $\theta_H = 0$ and $\theta_2 = 0.5$. It is recognized that situations with $\epsilon_1 \neq \epsilon_2$ and other values of θ_2 and θ_H can differ significantly from those considered in the present work. They will be investigated in the future.

Results show that the present solution method yields converging results very quickly. At each step of the successive approximations no more than 4 iterations are required for the solution of the nonlinear matrix. The number of iterations required for the higher-order approximation is even less since the previous-order result, which is already quite accurate, can be used as an initial guess. The successive approximation is carried out until two consecutive approximate solutions differ by less than 0.001 both in its prediction for the temperature profile and the radiative heat flux. In all considered cases, the above convergence criteria is achieved at no higher than the sixth-order approximation. Tables 1 and 2 illustrate the rate of convergence of the temperature profile for two typical cases. It is interesting to note the first-order result, which is generated by the solution of a simple 3×3 nonlinear matrix, already yields a resonably accurate description of the medium's temperature. The heat flux prediction is also quite satisfactory and superior to the common approach which treats the total heat flux as a sum of separate independent contributions from conduction and radiation. Utilizing a recently developed closed-form approximation for pure radiative transfer in an aniso-

Table 1 The first six-order approximate temperature profile θ with $N_1 = 0.01$, $L = 1.0$, $\epsilon_1 = \epsilon_2 = 0.5$ and $\omega_0 = 0$

τ/L	1st	2nd	3rd	4th	5th	6th
0.5	0.500	0.500	0.500	0.500	0.500	0.500
0.3	0.756	0.769	0.769	0.771	0.773	0.773
0.1	0.827	0.818	0.819	0.820	0.822	0.822
0.0	0.835	0.832	0.833	0.835	0.836	0.837
−0.1	0.842	0.845	0.846	0.848	0.849	0.849
−0.3	0.886	0.874	0.874	0.876	0.877	0.878
−0.5	1.000	1.000	1.000	1.000	1.000	1.000

tropically-scattering medium, the addition-solution suggests the following expression for the total heat flux

$$Q_t = \frac{4N_1}{L}(1-\theta_2) + \frac{1-\theta_2^4}{\dfrac{2}{\epsilon} - 1 + \left(\dfrac{3}{4} - \dfrac{\omega_0 x}{4}\right)L} \qquad (22)$$

A direct comparison between the first-order results, equation (22) and the higher order exact results obtained in the present work is shown in Table 3. It is apparent that the error of equation (22) can be quite substantial, particularly for cases with scattering and small emissivity.

Table 3 also shows that the present exact results differ slightly from the previously reported numerical results [7]. The discrepancy is not too unexpected since the previous computation was done nearly 20 years ago with a small IBM 650 computer [14]. The computer was slow and had a small core. Experience in the present calculation indicates that the accuracy of the exponential integral function. In a pure numerical computation with a small computer, slight error in the exponential integral function and consequently in the result of the calculation can be readily generated. The present work, on the other hand, utilizes a much larger and more efficient IBM 360 computer. All exponential integral functions are tabulated exactly up to eight significant figures using known analytical expressions and serves as the input of the calculation. The result should thus be more accurate and reliable.

The effect of the scattering albedo ω_0 and the forward-backward scattering parameter x on the medium's temperature profile is illustrated by Fig. 1, in which results of the temperature profiles for cases with $L = 1.0$, $\epsilon = 1.0$ and various values of N_1, ω_0 and x are presented. It readily demonstrates that scattering can have a significant effect on the medium's temperature. When $\omega_0 = 0$, the medium only absorbs and emits radiation. The interaction of conduction and radiation causes the temperature to rise above that of the pure conduction case. When $\omega_0 = 1.0$, the medium only scatters and does not either absorb or emit radiation. The temperature profile becomes linear. For intermediate values of ω_0, the temperature profile falls between the above two extreme cases. For $N_1 = 0.1$, the value of x has only a minor effect on the medium's temperature. As N_1 decreases, the effect of anisotropic scattering increases and is at a maximum at $N_1 = 0$. As expected, a strong forward scattering ($x = 1.0$) leads to an increase of the temperature near the cold wall and a decrease of the temperature near the hot wall. The opposite trend is observed for media with a strong backward scattering ($x = -1.0$). In all cases, the effect of x is quite negligible compared to the effect of ω_0.

The effect of scattering on heat transfer is illustrated by results presented in Tables 4 and 5. For the scattering albedo ω_0, its effect on heat transfer is most significant for systems with small N_1 and small surface emissivities. At $N_1 = 0.01$, $\epsilon = 0.1$ and $L = 1.0$, for example, the heat transfer result with $\omega_0 = 1.0$ represents a nearly 50 percent reduction from the result with $\omega_0 = 0$. At $\epsilon = 1.0$, on the other hand, the maximum variation of the total heat flux as ϵ_0 is changed from 0 to 1.0 is less than 10 percent. Unlike its almost negligible effect on the medium's temperature, the forward-backward scattering parameter also has a noticable effect on the heat transfer. But in contrast to the effect of ω_0, the effect of x is large for systems with large surface emissivity. At $\epsilon = 1.0$, $N_1 = 0.1$, $L = 1.0$ and $\omega_0 = 0.5$, for example, the total heat transfer in a strongly backward-scattering medium ($x = -1.0$) is about 10 percent less than that in a strongly forward-scattering medium ($x = 1.0$). At $\epsilon = 0.1$, however, the effect of x is almost negligible. At $L = 10.0$, the effect of ω_0 and x on the heat transfer results follows a similar pattern.

Physically, the above effect of ω_0 and x on the total heat transfer is not difficult to understand. For systems with small surface emissivity, energy leaving the hot bottom wall must travel many times across the medium before it is totally absorbed. There is an effective increase in the optical thickness. The effect of scattering is multiplied and thus becomes more apparent. The effect of ω_0 is therefore large for systems with small surface emissivity. But the multiple reflections by the two boundaries also cause a given ray of radiative energy to

Table 2 The first five-order approximate temperature profile θ with $N_1 = 1.0$, $L = 10.0$, $\epsilon_1 = \epsilon_2 = 1.0$, $\omega_0 = 0.5$ and $x = 1.0$

τ/L	1st	2nd	3rd	4th	5th
0.5	0.500	0.500	0.500	0.500	0.500
0.3	0.631	0.632	0.632	0.632	0.632
0.1	0.744	0.743	0.743	0.743	0.743
0.0	0.794	0.792	0.792	0.792	0.792
−0.1	0.840	0.838	0.837	0.837	0.837
−0.3	0.924	0.920	0.919	0.918	0.918
−0.5	1.000	1.000	1.000	1.000	1.000

Table 3 Comparison between the first-order result, equation (22), and the higher-order exact result. Values in parenthesis are numerical results reported in reference [6] ($L = 1$)

N_1	ω_0	x	First-order results equation (22)				Exact	
			$\epsilon = 1.0$	$\epsilon = 0.1$	$\epsilon = 1.0$	$\epsilon = 0.1$	$\epsilon = 1.0$	$\epsilon = 0.1$
1.0	0		2.569	2.202	2.536	2.047	2.572 (2.600)	2.221 (2.245)
	0.5	1.0	2.592	2.149	2.577	2.048	2.594	2.157
		0	2.549	2.145	2.536	2.047	2.550	2.154
		−1.0	2.511	2.142	2.500	2.047	2.512	2.150
0.1	0		0.765	0.371	0.736	0.247	0.769 (0.798)	0.402 (0.393)
	0.5	1.0	0.792	0.335	0.777	0.248	0.793	0.349
		0	0.748	0.333	0.736	0.247	0.750	0.346
		−1.0	0.711	0.330	0.700	0.247	0.712	0.343

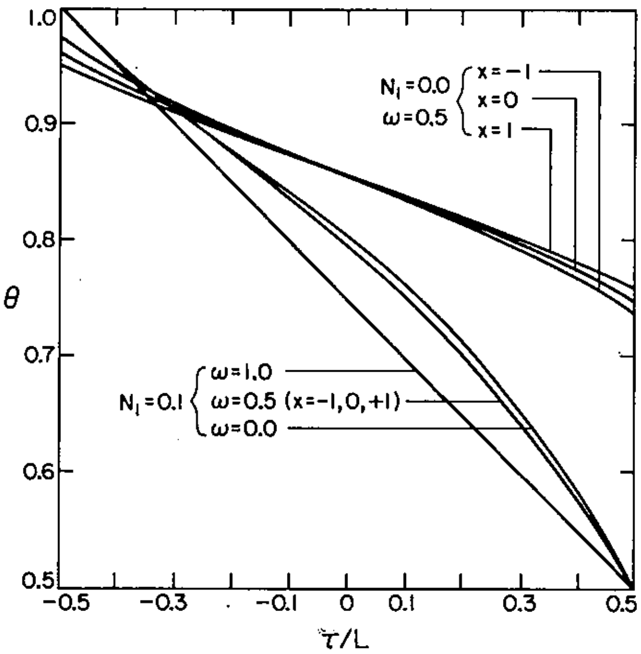

Fig. 1 The effect of anisotropic scattering on the medium's temperature with $L = 1.0$ and $\epsilon = 1.0$

Table 4 Effect of system parameters on the heat transfer result with $L = 1.0$

	N_1	$\epsilon_1 = \epsilon_2 = 1.0$			$\epsilon_1 = \epsilon_2 = 0.1$		
ω_0	x	1.0	0.1	0.01	1.0	0.1	0.01
0	0	2.572	0.769	0.567	2.221	0.403	0.158
	1.0	2.594	0.793	0.600	2.157	0.349	0.131
0.5	0	2.550	0.750	0.559	2.154	0.346	0.130
	−1.0	2.512	0.712	0.523	2.150	0.343	0.129
	1.0	2.602	0.802	0.622	2.048	0.248	0.068
1.0	0	2.519	0.719	0.539	2.047	0.247	0.067
	−1.0	2.456	0.656	0.476	2.047	0.247	0.067

Table 5 Effect of system parameters on the heat transfer result with $L = 10.0$

	M_1	$\epsilon_1 = \epsilon_2 = 1.0$			$\epsilon_1 = \epsilon_2 = 0.1$		
ω_0	x	1.0	0.1	0.01	1.0	0.1	0.01
0	0	20.115	2.115	0.315	20.105	2.106	0.305
	1.0	20.134	2.134	0.335	20.118	2.118	0.316
0.5	0	20.114	2.114	0.314	20.100	2.101	0.299
	−1.0	20.099	2.098	0.299	20.088	2.088	0.287
	1.0	20.155	2.155	0.355	20.039	2.039	0.239
1.0	0	20.110	2.110	0.310	20.035	2.035	0.235
	−1.0	20.085	2.085	0.285	20.032	2.032	0.232

totally change its direction repeatedly. After many reflections, the energy ray effectively "loses its sense of direction." The anisotropic effect of the scattering process becomes randomzied. The forward-backward scattering parameter x thus has only a relatively minor effect for systems with small surface emissivity.

Finally, it is interesting to note that for anisotropic-scattering media, an increase in the scattering albedo ω_0 does not necessarily lead to a reduction on the total heat transfer as it was suggested by the existing isotropic-scattering result [8] and confirmed by results shown in Tables 4 and 5 with $x = 0$. For many cases with $x \neq 0$, the total heat flux for a scattering medium ($\omega_0 \neq 0$) is actually greater than that of the corresponding nonscattering case ($\omega_0 = 0$). For cases considered in the present work, this unexpected behavior of anisotropic-scattering on heat transfer appears for systems with $\epsilon = 1.0$. Physically, this phenomenon can be explained by noting that a strong forward-scattering ($x = 1.0$) generally increases heat transfer. For systems with large surface emissivity in which the overall effect of ω_0 is small, the increase in heat transfer due to forward-scattering may, in some instances, be large enough to cause a net increase in the overall heat transfer. This rather "abnormal" effect of scattering does not appear for systems with $\epsilon = .1$. For those systems, the drop of the heat transfer due to the increase in ω_0 is large enough that even when $x = 1.0$, the heat transfer is less than that of the pure absorption case.

Concluding Remarks. The problem of simultaneous conduction and radiation through absorbing, emitting and anisotropically-scattering material is considered. The problem is solved by a successive approximation technique similar to the tradiational method of undetermined parameter. The conclusions that may be drawn from the presetn study are as follows:

1 The temperature profile of an anisotropically-scattering medium depends a great deal on the scattering albedo ω_0. The forward-backward scattering parameter x has only a minor effect on the medium's temperature.

2 Both ω_0 and x have important effects on the total heat transfer. The realtive importance of these effects depend on the surface emissivity of the boundaries. When ϵ is small, ω_0 has a significant effect on heat transfer and the effect of x is relatively unimportant. Scattering generally decreases heat transfer. When ϵ is large, on the other hand, the effect of ω_0 on the heat transfer is small and the effect of x becomes significant. The net heat transfer in a scattering medium can be greater or smaller than that of the pure absorption case depending on the value of x.

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